1 Precautionary Saving and Prudence

One behavior under uncertainty that is observed with regularity is the willingness to save more in the present in response to increased uncertainty in the future. This behavior is referred to as the *precautionary savings motive*. One might think that risk aversion can explain precautionary saving since it is a response to risk, but this is not true. Risk aversion only explains aversion to *contemporary risks*. The explanation of precautionary savings requires a concept that is related to risk aversion, but distinct from risk aversion. That concept is known as *prudence*.

This concept was first introduced by Hayne Leland with the following argument. Consider a two-period model where saving in the first period can be used to finance consumption in the second period. Assume that income in the first period, $y_1$, is certain, but income in the second period, $\tilde{y}_2$, is uncertain. Then the decision maker chooses first period saving, $s$, which earns rate of return, $r$, to solve the following problem:

$$
\max_s \quad U((1-s)y_1) + EU(\tilde{y}_2 + s(1+r)y_1)
$$

The first-order condition is:

$$
U'((1-s)y_1)y_1 = E\left(U'\left(\tilde{y}_2 + s(1+r)y_1\right)(1+r)y_1\right)
\Rightarrow
U'(c_1) = E(U'("\bar{c}_2"))(1+r) \quad (1)
$$

If the future income is certain, the first order condition for optimal saving is:

$$
U'(c_1) = U'(c_2)(1+r)
$$

Call the solution to this first order condition, $\bar{s}$ and corresponding consumption in Period 2, $\bar{c}_2$.

In order to analyze the savings response to increased uncertainty of future income, we expand both sides of the equation (1) about $(c_1, \bar{c}_2)$.

$$
U'(c_1) = U'(c_1)
E(U'(\bar{c}_2)) \approx U'("\bar{c}_2") + U''("\bar{c}_2")E(\bar{c}_2 - \bar{c}_2) + \frac{1}{2}U'''("\bar{c}_2")E((\bar{c}_2 - \bar{c}_2)^2)
\Rightarrow
U'(c_1) \approx U'("\bar{c}_2") (1+r) + \frac{1}{2}U'''("\bar{c}_2")\sigma_{\bar{c}_2}^2 (1+r)
$$

$U'$ is a decreasing function, so if $U''' > 0$ you must reduce $c_1$ and increase $c_2$ to restore the equality of the first order condition. In that case, saving in Period 1 is higher than in the case without uncertainty. We call the extra saving, precautionary saving.
Risk aversion alone does not explain precautionary saving. We need $U'' > 0$ to explain precautionary saving. This property of the utility function is referred to as prudence.

2 Example

In a two period model, suppose the agent’s income are $y_1$ and $y_2$ in period 1 and 2, respectively. In particular, $y_1 = y_2$. There is no initial wealth endowment. The agent’s problem is

$$\max_{c_1, c_2} U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

subject to

$$c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r}$$

The FOC is

$$u'(c_1) = (1 + r)\beta u'(c_2).$$

If $r = 0$ and $\beta = 1$, this condition would be reduced to

$$u'(c_1) = u'(c_2).$$

Then $c_1 = c_2 = y_1 = y_2$.

Now suppose there is uncertainty in agent’s income in period 2. The agent’s income in Period 2 can be $y^h_2$ and $y^l_2$ with equal probability. Moreover, $y^h_2 = y_1 + \sigma > 0$ and $y^l_2 = y_1 - \sigma > 0$ and then $\frac{y^h_2 + y^l_2}{2} = y_2 = y_1$. Specifically, we assume that $\sigma = \sqrt{\frac{3}{2}y_1}$

Then the agent’s problem is

$$\max_{c_1, c^h_2, c^l_2} U(c_1, c_2) = u(c_1) + \beta [0.5u(c^h_2) + 0.5u(c^l_2)]$$

subject to

$$c_1 + \frac{c^h_2}{1 + r} = y_1 + \frac{y^h_2}{1 + r}$$

$$c_1 + \frac{c^l_2}{1 + r} = y_1 + \frac{y^l_2}{1 + r}$$

The FOC is

$$u'(c_1) = (1 + r)\beta (0.5u'(c^h_2) + 0.5u'(c^l_2)).$$

If $r = 0$ and $\beta = 1$, this condition would be reduced to

$$u'(c_1) = 0.5u'(c^h_2) + 0.5u'(c^l_2).$$

Note that $\frac{c^h_2 + c^l_2}{2} = y_2 + y_1 - c_1$. If $u'$ is convex, that is $u'' > 0$, $0.5u'(c^h_2) + 0.5u'(c^l_2) > u'(y_2 + y_1 - c_1)$. So $u'(c_1) > u'(y_1 + y_1 - c_1)$ and the consumption $c_1$ in this problem would be lower than that in previous problem. If $u'$ is concave, that is $u'' < 0$, $0.5u'(c^h_2) + 0.5u'(c^l_2) <$
$u'(y_2 + y_1 - c_1)$. So $u'(c_1) < u'(y_1 + y_1 - c_1)$ and the consumption $c_1$ in this problem would be higher than that in previous problem.

If the utility function is log utility $\ln(c)$, the optimal consumption in period 1 would satisfy

$$2c_1^2 - 6y_1c_1 + 3y_1^2 + y_2^1 y_2^1 = 0.$$ 

The optimal consumption $c_1 = \frac{1}{4} \left( 6y_1 - \sqrt{(4y_1^2 + 8\sigma^2)} \right) = 0.5y_1 < y_1$ and therefore, saving in Period 1, $s > 0$. 