Solution to Exercise 1: Adverse Selection

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Question One: First Best

1. Suppose both the buyer and the seller know the realization of θ_i (i.e., no private information), what is the objective function of the buyer? What is the constraint the buyer faces?
   - The objective function for the principal (i.e., buyer) is

$$p[b(x_0) - w_0] + (1 - p)[b(x_1) - w_1] \tag{1}$$

   or

$$0.5(2x_0^{\frac{1}{2}} - w_0) + 0.5(2x_1^{\frac{1}{2}} - w_1)$$

   and the seller chooses (w_i, x_i).

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   and the seller chooses (w_i, x_i).
   - The constraints are

$$w_0 - \theta_0 x_0 = w_0 - x_0 \geq 0 \quad (PC0)$$

   and

$$w_1 - \theta_1 x_1 = w_1 - 2x_1 \geq 0 \quad (PC1).$$

Question One: First Best (Cont.)

1. What is the optimal contract (i.e., (w_i^{FB}, x_i^{FB}))?
   - Since the buyer knows θ_i, she can just maximize the total payoff (of the buyer and the seller) and extract all the payoff from the seller by setting $w_i^{FB} = \theta_i x_i^{FB}$.

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   - The optimization problem is

$$\max_{x_i} \quad p[b(x_0) - \theta_0 x_0] + (1-p)[b(x_1) - \theta_1 x_1].$$

The First Order Conditions (with respect to x_i's) are

$$x_i^{-\frac{1}{2}} = \theta_i \quad \textit{for } i \in \{0, 1\}.$$

Therefore, $x_0^{FB} = 1$ and $x_1^{FB} = \frac{1}{4}$.

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     Therefore, $x_0^{FB} = 1$ and $x_1^{FB} = \frac{1}{4}$.
   - Corresponding transfer payments are

     $$w_0^{FB} = \theta_0 x_0^{FB} = 1 * 1 = 1$$

     and

     $$w_1^{FB} = \theta_1 x_1^{FB} = 2 * 1/4 = \frac{1}{2}.$$

Question One: First Best (Cont.)

1. Which type of the agent has incentives to lie?
   - θ_0 type agent wants to lie.
   - In the state of θ_0, misreporting that the state is θ_1 generates a payoff of

$$w_1^{FB} - \theta_0 x_1^{FB} = 1/2 - 1 * 1/4 = \frac{1}{4} > 0.$$

Truthfully reporting yields a payoff of

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   - θ_1 type agent does *not* want to lie. In the state of θ_1, misreporting that the state is θ_0 generates a payoff of

     $$w_0^{FB} - \theta_1 x_0^{FB} = 1 - 2 * 1 = -1 < 0.$$

     Truthfully reporting yields a payoff of

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2. Economically, the agent wants to say my cost is high, even when the cost is low. This is because the agent gets paid much more when the cost is high.

Question Two: Second Best

1. The optimization problem:

$$
\begin{aligned}
\max_{w_i, x_i} \quad & p[b(x_0) - w_0] + (1-p)[b(x_1) - w_1] \\
s.t. \quad & w_0 - \theta_0 x_0 \geq w_1 - \theta_0 x_1, \qquad (IC0) \\
& w_1 - \theta_1 x_1 \geq w_0 - \theta_1 x_0, \qquad (IC1) \\
& w_0 - \theta_0 x_0 \geq 0, \qquad (PC0) \\
& w_1 - \theta_1 x_1 \geq 0. \qquad (PC1)
\end{aligned}
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2. Binding constraints (i.e., equalities): $IC0$ and $PC1$.
3. Non-binding constraints: $IC1$ and $PC0$.

Question Two: Second Best (Cont.)

1. We have two binding constraints:

$$w_1 = \theta_1 x_1 = 2x_1$$

and

$$w_0 = \theta_0 x_0 + w_1 - \theta_0 x_1 = \theta_0 x_0 + (\theta_1 - \theta_0)x_1 = x_0 + (2-1)x_1 = x_0 + x_1.$$

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2. Substituting the above two constraints into the objective function (in last slide), we arrive at

$$\max_{x_0, x_1} \quad 0.5[2x_0^{\frac{1}{2}} - x_0 - x_1] + 0.5[2x_1^{\frac{1}{2}} - 2x_1].$$

3. Notice that the principal needs to give the information rent, x_1, to θ_0 type agent.

Question Two: Second Best (Cont.)

1. The optimal contract is
   - The FOC with respect to x_0: $x_0^{-1/2} - 1 = 0 \rightarrow x_0^{SB} = 1 = x_0^{FB}$.
   - The FOC with respect to x_1: $-1 + x_1^{-1/2} - 2 = 0 \rightarrow x_1^{SB} = \frac{1}{9} < x_1^{FB}$.

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   - $w_1^{SB} = 2 * x_1^{SB} = \frac{2}{9} \rightarrow$ payoff to type 1 agent is $w_1^{SB} - 2 * x_1^{SB} = 2/9 - 2 * 1/9 = 0$.
   - $w_0^{SB} = x_0^{SB} + x_1^{SB} = 1 + \frac{1}{9} = \frac{10}{9} \rightarrow$ payoff to type 0 agent is $w_0^{SB} - x_0^{SB} = 10/9 - 1 = 1/9 > 0$.

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2. Type 0 receives a rent.
3. Type 1's output level is below the First Best level.

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2. Type 0 receives a rent.
3. Type 1's output level is below the First Best level.
4. Tradeoff between achieving allocative efficiency and reducing the information rent.