1. *The Solow Model with both Population Growth and Technological Progress.* Assume that there is only a representative agent in the economy. The Solow model can be formulated as follows:

$$\begin{aligned} K_{t+1} &= (1-\delta) K_t + sF(K_t, N_t), \quad (1) \\ A_{t+1} &= (1+g_A) A_t, \\ N_{t+1} &= (1+g_N) N_t, \end{aligned}$$

given K_0, A_0, and N_0, where $F(K_t, N_t) = A_t K_t^{\alpha} N_t^{1-\alpha}$.

a) Define $k = \frac{K}{A^{1/(1-\alpha)}N}$ and $y = \frac{Y}{A^{1/(1-\alpha)}N}$, show that the dynamic equation about capital per efficient labor unit (k) takes the following form:

$$k_{t+1} = \frac{(1-\delta) k_t + sf(k_t)}{(1+g_A)^{1/(1-\alpha)}(1+g_N)}. \quad (2)$$

where $f(k_t) = k_t^{\alpha}$.

Answer: Dividing $A_{t+1}^{1/(1-\alpha)} N_{t+1}$ on both sides of (1), we have

$$\begin{aligned} \frac{K_{t+1}}{A_{t+1}^{1/(1-\alpha)} N_{t+1}} &= (1-\delta) \frac{A_t^{1/(1-\alpha)} N_t}{A_{t+1}^{1/(1-\alpha)} N_{t+1}} \frac{K_t}{A_t^{1/(1-\alpha)} N_t} + s \frac{A_t^{1/(1-\alpha)} N_t}{A_{t+1}^{1/(1-\alpha)} N_{t+1}} \frac{A_t K_t^{\alpha} N_t^{1-\alpha}}{A_t^{1/(1-\alpha)} N_t}, \Longrightarrow \\ k_{t+1} &= \frac{(1-\delta)}{(1+g_A)^{1/(1-\alpha)}(1+g_N)} k_t + \frac{s}{(1+g_A)^{1/(1-\alpha)}(1+g_N)} k_t^{\alpha} \Longrightarrow \\ k_{t+1} &= \frac{(1-\delta) k_t + sf(k_t)}{(1+g_A)^{1/(1-\alpha)}(1+g_N)}. \end{aligned}$$

b) In the steady state, find the expressions of (1) capital stock per effective worker, k^*, and (2) consumption per effective worker, c^*, in terms of model parameters.

Answer: In the steady state,

$$k^* = \frac{(1-\delta) k^* + s(k^*)^{\alpha}}{(1+g_A)^{1/(1-\alpha)}(1+g_N)} \Longrightarrow$$

$$\left[(1+g_A)^{1/(1-\alpha)}(1+g_N) - (1-\delta)\right] k^* = s(k^*)^{\alpha} \Longrightarrow$$

$$k^* = \left[\frac{s}{(1+g_A)^{1/(1-\alpha)}(1+g_N) - (1-\delta)}\right]^{1/(1-\alpha)}$$

and

$$\begin{aligned} C_t &= (1-s) F(K_t, N_t) \Longrightarrow \\ c_t &= (1-s) k_t^{\alpha} \Longrightarrow c^* = (1-s)(k^*)^{\alpha}. \end{aligned}$$

c) On the balanced growth path, find (1) the growth rate of output per effective worker $\left(\frac{Y}{A^{1/(1-\alpha)}N}\right)$ and (2) the growth rate of output per worker $\left(\frac{Y}{N}\right)$. Which of the above three growth rates is a measure for the standards of living in the economy?

Answer: (1) The growth rate of output per effective worker is

$$g_y = 0. \tag{3}$$

(2) the growth rate of output per worker is

$$g_{Y/N} = \frac{1}{1-\alpha} g_A. \tag{4}$$

$g_{Y/N}$ is a measure for the standards of living in the economy.

d) What is the golden rule level of k_G that maximizes the steady state level of consumption per effective worker (c^*), i.e., $\frac{dc^*}{dk^*} = 0$ for $k^* = k_G$? Note that here you may need to use this approximation: $(1+g_A)^{1/(1-\alpha)}(1+g_N) = 1 + \frac{1}{1-\alpha} g_A + g_N$.

Answer: Given that

$$\begin{aligned} c^* &= (1-s)(k^*)^\alpha \\ &= (k^*)^\alpha - s(k^*)^\alpha \\ &= (k^*)^\alpha - \left(\frac{1}{1-\alpha} g_A + g_N + \delta\right) k^* \end{aligned}$$

where we use the facts that in the steady state $(1+g_A)^{1/(1-\alpha)}(1+g_N) = 1 + \frac{1}{1-\alpha} g_A + g_N$ and $\left(\frac{1}{1-\alpha} g_A + g_N + \delta\right) k^* = s(k^*)^\alpha$. Therefore, the optimal condition is:

$$\begin{aligned} \frac{dc^*}{dk^*} &= \alpha (k^*)^{\alpha-1} - \left(\frac{1}{1-\alpha} g_A + g_N + \delta\right) = 0 \Longrightarrow \\ k^* &= \left(\frac{\alpha}{\frac{1}{1-\alpha} g_A + g_N + \delta}\right)^{1/(1-\alpha)}. \end{aligned} \tag{5}$$