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Note on a Two-Period Optimal Consumption-Saving Model

1. Consider an individual consumer's optimal consumption and saving problem: The consumer lives for two periods: young and old periods, and receives disposable income in these two periods. The consumer chooses optimal consumption to maximize his or her lifetime utility:

$$\max_{\{c,c^o\}} \frac{c^{1-1/\sigma}-1}{1-1/\sigma} + \beta\left(\frac{(c^o)^{1-1/\sigma}-1}{1-1/\sigma}\right), \tag{1}$$

where $\frac{c^{1-1/\sigma}-1}{1-1/\sigma}$ and $\frac{(c^o)^{1-1/\sigma}-1}{1-1/\sigma}$ are the utility functions when the consumer is young and old, respectively, subject to

$$c + \frac{c^o}{R} = a + y + \frac{y^o}{R}, \tag{2}$$

where $\sigma > 0$ measures the elasticity of intertemporal substitution, β is the discount factor, R is the gross interest rate, c and c^o are consumption in young and old periods, respectively, a is financial wealth, and y and y^o are given disposable income received in young and old periods, respectively.

2. We can solve this problem using the following method. First, using the budget constraint to rewrite the expression for c^o:

$$c^o = R\left(a + y + \frac{y^o}{R} - c\right). \tag{3}$$

Second, substituting this expression into the objective function yields:

$$\max_{c} \frac{c^{1-1/\sigma}-1}{1-1/\sigma} + \beta\left(\frac{\left(R\left(a+y+\frac{y^o}{R}-c\right)\right)^{1-1/\sigma}-1}{1-1/\sigma}\right). \tag{4}$$

Taking derivative with respect to c leads to the following optimal condition for c:

$$c^{-1/\sigma} - \beta R\left(R\left(a+y+\frac{y^o}{R}-c\right)\right)^{-1/\sigma} = 0.$$

Solving for c yields:

$$c^* = \frac{1}{1+\beta^\sigma R^{\sigma-1}}\left(a+y+\frac{y^o}{R}\right).$$

Finally, combining it with the budget constraint yields:

$$(c^o)^* = \frac{(\beta R)^\sigma}{1+\beta^\sigma R^{\sigma-1}}\left(a+y+\frac{y^o}{R}\right). \tag{5}$$

Given the expression of c^*, the expression for optimal saving in the young period can be written as

$$s^* \equiv y - c^* = \frac{\beta^\sigma R^{\sigma-1}}{1+\beta^\sigma R^{\sigma-1}} y - \frac{1}{1+\beta^\sigma R^{\sigma-1}} \left(a + \frac{y^o}{R} \right),$$

which means that saving increases with the interest rate when $\sigma > 1$. It is worth noting that when $\sigma = 1$, the power function, $\frac{c^{1-1/\sigma}-1}{1-1/\sigma}$, will converge to the log utility, i.e.,

$$\lim_{\sigma \to 1} \frac{c^{1-1/\sigma} - 1}{1 - 1/\sigma} = \ln(c). \tag{6}$$

In this case,

$$c^* = \frac{1}{1+\beta} \left(a + y + \frac{y^o}{R} \right), (c^o)^* = \frac{\beta R}{1+\beta} \left(a + y + \frac{y^o}{R} \right), s^* = \frac{\beta}{1+\beta} y - \frac{1}{1+\beta} \left(a + \frac{y^o}{R} \right). \tag{7}$$