Demographic structure and capital accumulation: A quantitative assessment

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ABSTRACT


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1. Introduction

Motivated by the lack of robust evidence of demographic impact on asset prices such as the interest rate (Poterba, 2001), d’Albis (2007) develops an analytically tractable overlapping-generations model with life-cycle saving motive and a general mortality pattern. He shows that the effect of a change in the population growth rate (through fertility change) on capital accumulation may be positive or negative. The result contrasts sharply to the predicted negative relation between population growth and capital accumulation in two most well-known overlapping-generations models – the two-period overlapping-generations model (Diamond, 1965) and the continuous-time overlapping-generations model with age-invariant mortality (Blanchard, 1985).

The approach in d’Albis (2007) is mainly theoretical, and he uses a model with age-specific mortality rates, a general production function with no technological progress, and age-varying discount rates. Moreover, d’Albis (2007) follows Blanchard (1985) in assuming that the agents do not retire. To deliver sharper predictions, this paper complements his analysis by providing quantitative assessment regarding the relation between fertility change and capital accumulation in industrial countries. As such, this paper uses empirical age-specific mortality rates of USA and a constant discount rate, allows for technological progress, and chooses as values of production and preference parameters those commonly used in other studies. Two versions of the overlapping-generations model are considered in this paper. First, this paper examines a...
model without retirement, so as to facilitate comparison with d’Albis (2007). It then incorporates the empirically relevant feature of retirement and considers the case in which the workers retire.

Two major conclusions arise from the quantitative exercises. First, when the population growth rate increases from, say, −6% to 2% in the model with no retirement, its effect on the steady-state capital is first positive and then negative. This result gives some support to the general prediction in d’Albis (2007). Second, when the population growth rate changes within the more relevant interval from about −2% to 2%, the effect on capital accumulation is monotonically negative in both versions of the model (with or without retirement). Overall, the results of this paper suggest that while a positive relation between population growth and capital accumulation is possible, their relation is practically always negative for the parameters relevant for industrial countries. In this sense, the prediction of the highly stylized overlapping-generations models of Diamond (1965) and Blanchard (1985) regarding the sign of this relation is not misleading.

The paper is organized as follows. Section 2 discusses the model. Section 3 characterizes the steady-state equilibrium of the model, and discusses the existence and uniqueness of the equilibrium. Section 4 examines the effects of fertility changes on steady-state capital accumulation in an environment without retirement. Section 5 examines the same issues in an environment with retirement. Section 6 discusses how the model can be extended to examine the impact of social security changes. Section 7 provides concluding remarks.

2. The model

This paper considers a continuous-time overlapping-generations model, modified from Tobin (1967), Blanchard (1985), Bommier and Lee (2003), and d’Albis (2007). This section gives a brief description of the model; a more detailed discussion can be found in the above papers, especially d’Albis (2007). The demographic and labor supply features of the model will be discussed first, to be followed by a discussion of the consumption decision of individual cohorts.

Represent the probability that an individual survives to at least age \( x \) by the survival function \( l(x) \), where \( x \in [0, \Omega] \). \( \Omega \) is the maximum age, and \( l(0) \) is normalized to 1. Since most overlapping-generations models assume that the adult stage starts at age 20, in the following analysis \( x \) is interpreted as adult age, which is defined as actual age minus 20. Using the life table information for the USA (men and women combined) in 2004, obtained from the Human Mortality Database (2007), one can calculate the survival probability in terms of adult age according to \( l(x) = l_{\text{actual}}(x + 20)/l_{\text{actual}}(20) \), where \( l_{\text{actual}}(\cdot) \) is the survival probability based on actual age.\(^2\) The survival function \( l(x) \) in USA 2004 is shown in Fig. 1.

The instantaneous mortality rate at age \( x \), \( \mu(x) \), is related to \( l(x) \) by

\[
\mu(x) = -\frac{1}{l(x)} \frac{dl(x)}{dx}.
\]

In the following analysis, the age-specific mortality rates are assumed to be time-invariant and \( \Omega \) is taken to be 90 (since maximum age is 110 in the US life table).

It is also assumed, implicitly, that age-specific fertility rates are time-invariant. However, since fertility decisions are usually not modelled in the framework used in Blanchard (1985) and d’Albis (2007), this paper simply uses the alternative,\(^2\) It is more convenient to express age as a continuous variable in the following analysis, even though the life table data are available only for discrete years of age (and thus, the quantitative exercises are based on discrete years of age). Moreover, while it is more appropriate to use cohort survival information, this paper uses a period life table because cohort life tables of people born recently are not available.
but equivalent, assumption that the growth rate of births is constant.\(^3\) That is,
\[
B(t) = B(0)e^{\alpha t},
\]  
where \(B(t)\) is the number of births at time \(t\), and \(n\) is the constant growth rate of the number of births.

Based on the birth and death histories of different cohorts, the population at time \(t\) is given by
\[
P(t) = \int_{t-\Omega}^{t} B(s)l(t-s)\, ds = \int_{t-\Omega}^{t} B(t)e^{-\nu(t-s)}l(t-s)\, ds = B(t)\int_{0}^{\Omega} e^{-\nu x}l(x)\, dx.
\]  
The term \(\int_{0}^{\Omega} e^{-\nu x}l(x)\, dx\) in (3) represents the sum of the number of survived persons (when \(l(0)\) is normalized to 1) at various ages, adjusted by \(n\) (growth rate of births). Since this term is independent of time \(t\), it is clear that the population grows at the same rate as that of the number of births.

The labor supply and retirement assumptions used in this paper are given as follows. Since these features are not the major focus in the current investigation, this paper follows Blanchard (1985) and d'Albis (2007) to assume that these decisions are exogenous. In the Blanchard (1985) model with infinite \(\Omega\) and the d'Albis (2007) model with finite \(\Omega\), there is no retirement.\(^4\) This assumption is followed in one version of the model used in this paper. In another version of the model, which takes retirement into account, it is assumed that individuals work up to an age \(T_r\) strictly less than \(\Omega\) and then retire. Thus, the labor supply specification in both versions of the model can be represented by
\[
N(s, v) = \begin{cases} 
1 & \text{if } 0 \leq v - s \leq T_r, \\
0 & \text{if } T_r < v - s \leq \Omega,
\end{cases}
\]  
where \(N(s, v)\) is the (exogenous) labor supply of a cohort \(s\) individual at time \(v\), and \(T_r\) is the retirement age. Note that \(T_r = \Omega\) in the first version of the model and \(T_r < \Omega\) in the second version.

Next, consider individual’s consumption decision. An individual born at time \(s\) chooses \(C(s, v)\) at time \(t\) (where \(s \leq t \leq s + \Omega\)) to maximize
\[
\int_{s}^{s+\Omega} e^{-\rho(v-t)} \left[ \frac{C(s, v)^{1-1/\sigma}}{1-1/\sigma} \right] \, dv,
\]  
subject to the flow budget constraint
\[
\frac{dZ(s, v)}{dv} = \left[ r(v) + \mu(v - s)Z(s, v) + w(v)N(s, v) - C(s, v) \right],
\]  
where \(\rho\) is the discount rate, \(\sigma\) is the intertemporal elasticity of substitution, \(C(s, v)\) is consumption of a cohort \(s\) individual at time \(v\), \(Z(s, v)\) is the financial wealth of a cohort \(s\) individual at time \(v\), \(r(v)\) is the (real) interest rate at \(v\), and \(w(v)\) is the (real) wage rate at \(v\). Individuals are born without financial assets or liabilities, and face a terminal condition of non-negative financial wealth. These two boundary conditions for a particular cohort are given by
\[
Z(s, s) = 0, \quad Z(s, s + \Omega) \geq 0.
\]

A brief interpretation of (5) and (6) is as follows. An individual's lifetime satisfaction is given by the present discounted value of a stream of utilities, where the instantaneous utility function is of the constant-intertemporal-elasticity-of-substitution (CIES) form. To focus purely on the saving-for-retirement motive, this paper follows Blanchard (1985) and d'Albis (2007) to assume that individuals have no bequest motive. Note that the objective function in (5) is weighted by the conditional probability of survival at different ages, \(l(v-s)/l(t-s)\). The budget constraint (6) assumes the presence of an actuarially fair annuity (Yaari, 1965), in which an individual of age \(x\) will surrender all the financial wealth \(Z(s, s + x)\) to the insurance company if death occurs, but will receive an extra amount equal to \(\mu(x)Z(s, s + x)\) if death does not occur.\(^5\) The terms of the annuity contract change as an individual ages, reflecting increasing mortality rates, and this is a natural extension of the assumption made in Blanchard (1985).

It can be shown that the Keynes–Ramsey rule for this intertemporal consumption problem is that for \(v \geq t,\)
\[
\frac{dC(s, v)}{dv} = \sigma\left[ r(v) - \rho \right]C(s, v).
\]  

---

\(^3\) Note that the growth rate of births is time-invariant when age-specific birth and death rates are unchanged for different cohorts, according to well-known results in stable population theory introduced by Lotka (1939). (See Keyfitz and Caswell, 2005, for example, for more detailed discussion about the stable population theory.)

\(^4\) Blanchard (1985, Section III) has an alternative specification by introducing exponentially decreasing labor income through life, an assumption which is not quite empirically relevant. Similarly, d'Albis (2007, Section 3.3) specifies an exponentially declining labor income process as an individual ages, but he interprets it as an intergenerational transfer scheme.

\(^5\) Note that while the actuarially fair financial contract may not literally exist in an economy, this tractable analytical device captures the fact that individuals take mortality risk into account in making consumption decisions.
As in Blanchard (1985) and d’Albis (2007), this paper considers a closed economy and assumes a standard neoclassical production function. To carry out quantitative exercises relevant to a growing economy, this paper assumes a Cobb–Douglas production function with exogenous technological progress, which is given by

\[ Y(t) = F(K(t), A(t)N(t)) = K(t)^{1-z}A(t)N(t)^{1-z}, \]

where \(0 < z < 1\), and \(Y(t), K(t), N(t)\) and \(A(t)\) represent, respectively, output, capital input, labor input and technological level at time \(t\). Technological progress is represented by

\[ A(t) = A(0)e^{\theta t}, \]

where \(g\) is the rate of labor-augmenting technological progress. Capital accumulates according to

\[ \frac{dK(t)}{dt} = Y(t) - C(t) - \delta K(t), \]

where \(\delta\) is the depreciation rate.

Since labor supply is exogenous, it can be shown from (2) and (4) that

\[ N(t) = \int_{-\infty}^{t} B(s)N(t-s)N(s,t)ds = B(t) \int_{0}^{T} e^{-\lambda(x)}dx. \]

Therefore, aggregate labor supply grows at the same rate (i.e., \(n\)) as the population.

To summarize, this paper considers an overlapping-generations model with age-specific mortality rates. In sharp contrast to the assumption of age-invariant mortality rates used in Blanchard (1985), age-specific mortality rates are used in d’Albis (2007) and this paper. However, there are differences between d’Albis (2007) and this paper. In d’Albis (2007), there is no technological progress, the mortality pattern and the production function are very general, and discount rates may be age-varying. Moreover, d’Albis (2007) assumes that there is no retirement. This paper generalizes d’Albis (2007) by allowing for technological progress, but uses empirical age-specific mortality rates of USA, Cobb–Douglas production function and a constant discount rate. One version of the model considered in this paper also specifies a retirement stage. These features are more relevant for quantitative assessment of the theoretical model.

3. The steady-state equilibrium

Define a variable per unit of effective labor as the variable divided by \(AN\), and denote it in lower case letter (such as \(y(t) = Y(t)/A(t)N(t)\)). With this transformation, the production function in intensive form is given by

\[ y(t) = F \left( \frac{K(t)}{A(t)N(t)} \right)^{1-z} = f(k(t)) = k(t)^{\gamma}. \]

Moreover, (10)-(13) lead to

\[ \frac{dk(t)}{dt} = k(t)^{\gamma} - c(t) - (\delta + g + n)k(t). \]

Now, consider the steady-state equilibrium of the economy. Denote a variable at the steady-state equilibrium with a * (e.g., \(k^*\) is the steady-state value of capital per unit of effective labor). The steady-state equilibrium is defined by \(dk(t)/dt = 0\) in (14); that is,

\[ (k^*)^{\gamma} - c^* - (\delta + g + n)k^* = 0. \] (14a)

Furthermore, the competitively determined interest rate and wage rate at the steady-state equilibrium are given by

\[ r^*(t) = f'(k^*) - \delta = \alpha(k^*)^{\gamma-1} - \delta \equiv r^* \]

and

\[ w^*(t) = A(t)f(k^*) - k^*f'(k^*) = A(t)(1 - \alpha)(k^*)^{\gamma} \equiv A(t)w^*. \] (16)

To characterize the steady-state equilibrium, it is useful to express \(c^*\) in terms of \(k^*\), so that \(k^*\) can be solved using (14a). It is more transparent to break the derivation into several steps. First, use the lifetime budget constraint to express the starting consumption level of a particular cohort in terms of other variables. Integrating (8) along the steady-state equilibrium gives

\[ C^*(s,v) = e^{S^*(v-s)}C^*(s,v), \]

where \(C^*(s,s)\) is the steady-state consumption of a cohort \(s\) individual at the beginning of (adult) life, and

\[ g^*_c = \sigma(r^* - \rho) \] (18)
is the steady-state growth rate of individual consumption. Substituting \((4), (7), (10)\) and \((15)-(18)\) into a cohort \(s\) individual’s lifetime budget constraint at the beginning of adult life, and then simplifying, yields \(^6\)

\[
C^*(s, s) = e^{s}w^*A(0) \frac{\int_{0}^{T} e^{-\eta r - \xi^s y(x)} dx}{\int_{0}^{T} e^{-\eta r - \xi^s y(x)} dx}.
\]

(19)

Since all terms on the right-hand side, except \(e^s\), of (19) are independent of \(s\), it is easy to conclude that starting consumption levels of different cohorts are growing at \(g\), the rate of technological progress. \(^7\)

Second, aggregate consumption at the steady-state equilibrium is related to the starting consumption level of a particular cohort, \(C^*(0, 0)\), according to

\[
C^*(t) = \int_{t}^{T} B(s)(t - s)C^*(s, t) ds
\]

\[
= \int_{t}^{T} B(t) e^{-m(t-s)}(t-s)e^s(t-s)C^*(s, s) ds
\]

\[
= B(t)e^t C^*(0, 0) \int_{0}^{T} e^{-g(t-n)\xi^s y(x)} dx.
\]

(20)

In (20), \(C^*(s, t)\) of an arbitrary cohort at the current time is first expressed in terms of \(C^*(s, s)\) at the beginning of adult life, and then \(C^*(s, s)\) of cohort \(s\) is expressed in terms of \(C^*(0, 0)\) of an earlier cohort.

Finally, combining the above results, consumption per unit of effective labor can be expressed in terms of capital per unit of effective labor according to \(^8\)

\[
k^* = \frac{C^*(t)}{A(t)N(t)} = (1 - z)(k)^z \left( \int_{0}^{T} e^{-\eta r - \xi^s y(x)} dx \right) \left( \int_{0}^{T} e^{-g(t-n)\xi^s y(x)} dx \right).
\]

Substituting (21) into (14a) yields the equation that determines capital per unit of effective labor at the steady-state equilibrium: \(^9\)

\[
(k)^z - (1 - z)(k)^z \left( \int_{0}^{T} e^{-\eta r - \xi^s y(x)} dx \right) \left( \int_{0}^{T} e^{-g(t-n)\xi^s y(x)} dx \right) = (\delta + g + n)k^*.
\]

(22)

Note that \(r^*\) and \(g^*\) depend on \(k^*\) only, according to (15) and (18). Eq. (22) has an interpretation familiar in the neoclassical growth model \((\text{Solow, 1956; Cass, 1965})\), with the actual investment per unit of effective labor on the left-hand side and the break-even investment per unit of effective labor \((i.e.,\) the amount of current investment per unit of effective labor which is required to keep a constant level of capital per unit of effective labor) on the right-hand side.

The above derivation of the steady-state equilibrium condition, while somewhat tedious, provides an alternative to the approach used in d’Albis (2007). It differs from d’Albis (2007) in two aspects – derivation and interpretation. First, the derivation in this paper is arguably more transparent, since it proceeds with familiar economic reasoning (expressing consumption of an individual at the beginning of adult life in terms of other variables in the lifetime budget constraint) in (19) and the aggregation formula (expressing aggregate consumption as a weighted average of different cohorts’ consumption levels) in (20). Second, d’Albis (2007) characterizes the steady-state equilibrium in terms of the fixed point of a mathematical function (see (19) of that paper), but (22) of this paper expresses the equilibrium in terms of familiar

\(^6\) Integrating the flow budget constraint (6) to give the lifetime budget constraint at time \(t\) as

\[
\int_{t}^{T} e^{-\eta r - \xi^s y(x)} dy N(s, v) dv = Z(s, t) + \int_{t}^{T} e^{-\eta r - \xi^s y(x)} dw N(s, v) dv.
\]

(6a)

Substituting \((4), (7), (10)\) and \((15)-(18)\) into (6a) at the beginning of adult life \((t = s)\) yields

\[
\int_{t}^{T} e^{-\eta r - \xi^s y(x)} dy N(s, s) dv = \int_{t}^{T} e^{-\eta r - \xi^s y(x)} dw N(s, s) dv.
\]

Finally, the relation between survival probability and mortality rate in (1) can also be expressed as

\[
k(x) = e^{-\int_{0}^{T} \mu^s dx}.
\]

(1a)

\(^7\) Therefore, starting consumption levels of different cohorts are related by

\[
C^*(s, s) = e^{s}C^*(0, 0).
\]

(19a)

\(^8\) As seen from (19), the ratio \(\frac{\int_{0}^{T} e^{-\eta r - \xi^s y(x)} dx}{\int_{0}^{T} e^{-\eta r - \xi^s y(x)} dx}\) comes from an individual cohort’s intertemporal constraints, and these two integrals represent longitudinal constraints. The other two integrals in (21) are cross-sectional constraints, corresponding to summation (of consumption and labor input, respectively) across different cohorts at a particular time.

\(^9\) Eq. (22) can be simplified by eliminating \((k)^z\) from both sides. However, the current form of (22) is kept because its economic interpretation can be seen more clearly.
economic concepts of actual and break-even investment. While these two characterizations are equivalent, the equation used in this paper is easier to interpret for most economists.

Proposition 1 states the sufficient condition under which the steady-state equilibrium exists and is unique. The proof is given in Appendix A.

**Proposition 1.** For the overlapping-generations model with age-specific mortality rates, no labor supply during retirement, CIES utility function and Cobb–Douglas production function, the steady-state equilibrium (with \(k' > 0\)) exists and is unique, if

\[-\sigma^2 h'(-\sigma u + g + n + \sigma \rho) + (1 - \sigma) h'(1 - \sigma)u + \sigma \rho) - h'(u - g) > 0\]  

for all \(u \in (-\delta, \infty)\), where \(h'(\cdot)\) is the derivative of \(h(\cdot)\) in (A.4) and \(h''(\cdot)\) is the derivative of \(h'(\cdot)\) in (A.5).

4. Quantitative assessment: no retirement

\(^{10}\) Note that when \(g = 0\), (A.7) in Appendix A is the same as (19) of \(d'\text{Albis} (2007)\) if the production function is Cobb–Douglas.

\(^{11}\) Note that condition (23) is always satisfied when \(\sigma = 1\); see Lemma 8 in \(d'\text{Albis} (2007)\). We have also checked that it is satisfied for the robustness analysis (with \(\sigma = 0.5\)) in Fig. 4.

\(^{12}\) \(B\text{arro}\) et al. (1995) use \(\sigma = 0.5\) in their baseline specification, and consider \(\sigma = 1\) in their robustness analysis. The conclusions of this paper hold irrespective of whether \(\sigma = 0.5\) or 1 is used. One reason of choosing \(\sigma = 1\) in the baseline specification is that the upward-sloping region in the relation between \(n\) and \(k'\) can be seen more clearly; compare Figs. 2 and 4.

\(^{13}\) The range of population growth rate from \(-2\%\) to \(2\%\) is adequate to cover the experience of most economies. Because the sign of the relation between \(k'\) and \(n\) changes when \(n\) is about \(-4.4\%\) in the baseline specification of one version of the model (see Fig. 2), this paper presents the results for \(n\) ranging from \(-6\%\) to \(2\%\).

\(^{14}\) Appendix B provides a decomposition of the overall change into these two effects.
the capital dilution effect, which is clearly illustrated in the Solow model with constant saving rate. However, unlike the Solow model, the aggregate saving rate in an overlapping-generations model may change endogenously when the population grows at a different rate. As shown in row 2 of Table 1, when the population growth rate increases in the overlapping-generations model with US age-specific mortality rates, aggregate saving rate increases as a result. In principle, the sign of the effect of a change in \( n \) on \( k^* \) is ambiguous, depending on the relative magnitudes of the capital dilution and saving effects (see the discussion in d’Albis, 2007, p. 411 also). The calculations based on the baseline specification (as illustrated by rows 3 and 4 in Table 1) show that while these two effects are opposing, the saving effect is dominated by the capital dilution effect for a wide range of population growth rates, including \(-2\% \leq n \leq 2\%\).

It is helpful to further investigate how the aggregate saving rate changes as a result of a rise in the population growth rate. Since aggregate saving is the weighted average of individual cohorts’ savings, one way to understand the change in aggregate saving is to see whether the change is mainly due to individual saving behavior or due to the proportion of savers in the economy. In an overlapping-generations model with purely life-cycle saving motive, a change in fertility does not affect the variables relevant for an individual’s consumption problem – in terms of utility function (5) and budget constraint (6) – directly. The main effect on capital accumulation is not due to individual saving behavior, but is due to the remaining factor of a change in the composition of savers versus dissavers. When the population growth rate is low (say, at \(-6\%) initially, there is a high proportion of old dissavers and aggregate saving rate is low. As a result, a rise in population growth increases the proportion of young savers and this effect is relatively significant, causing the saving effect, given by the ratio in (B.5), to be larger and more likely to dominate the capital dilution effect. On the other hand, when the population growth rate is high (say, at 1%) initially, aggregate saving rate is high and thus, the saving effect is more likely to be smaller and be dominated by the capital dilution effect.

It remains to assess the robustness of the results in Fig. 2 when the parameter values change. In particular, if the hump-shaped curve in Fig. 2 is shifted significantly to the right when the parameter values are different, then the conclusion that the relevant range of population growth rates (from, say, \(-2\% \) to \(2\%\)) is in the decreasing portion of the curve may be reversed.

Among the various parameters of the model, robustness analyses regarding the production and preference parameters are widely regarded as the most interesting ones. First, consider different values of the production parameters. The values of other parameters are kept at their baseline values. Besides the value of 0.3 used in Barro et al. (1995) and this paper, the capital share (\( \alpha \)) is taken to be about \( \frac{1}{2} \) by many researchers, and an even higher value of 0.42 has been used in King et al. (1988). Fig. 3 illustrates the relationship between \( n \) and \( k^* \) when \( \alpha = \frac{1}{2} \). It is observed that the hump-shaped curve shifts leftward and the downward-sloping segment starts at about \(-4.6\%\). When the capital share varies in the range of 0.3–0.42, it is found that the relationship between population growth and steady-state capital is always hump-shaped, and it shifts to the left when the capital share increases.

Next, consider different values of the intertemporal elasticity of substitution parameter (\( \sigma \)). In the literature, researchers usually take \( \sigma \) as a value smaller than or equal to 1. We try different values of \( \sigma \) and find that the relation between \( n \) and \( k^* \) shifts to the left when \( \sigma \) decreases. Fig. 4 presents the relationship when \( \sigma = 0.5 \), which has been commonly used in other papers, such as Barro et al. (1995).

We also preform robustness analysis for the discount rate from 0 to 0.04. It is found that the relation between population growth rate and steady-state capital is still hump-shaped and the curve is shifted to the right when the discount rate decreases. Fig. 5 presents the relation between these two variables when the discount rate is zero. In this case, the downward-sloping segment starts at about \(-3.7\%\).

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**Notes:** The capital dilution effect and saving effect are calculated according to (B.4) and (B.5), respectively, where \( x = 0.3, \delta = 0.05, g = 0.02, \) the population growth rate \( (n) \) and aggregate saving rate \( (s) \) in the current column refer to the initial equilibrium, and the corresponding entries in the next column refer to the final equilibrium. For example, 1/2.6918 refers to the capital dilution effect when \( n_1 = -0.06 \) and \( n_2 = -0.05, \) and is calculated according to (B.4). Note that the overall effect is positive (i.e., the saving effect dominates the capital dilution effect) if the product of the capital dilution and saving effects is larger than 1.

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**Table 1**

Population growth, aggregate saving and capital accumulation (with no retirement)

<table>
<thead>
<tr>
<th>Population growth rate, ( n )</th>
<th>(-6%)</th>
<th>(-5%)</th>
<th>(-4%)</th>
<th>(-3%)</th>
<th>(-2%)</th>
<th>(-1%)</th>
<th>0</th>
<th>1%</th>
<th>2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate saving rate, ( s )</td>
<td>(%)</td>
<td>2.989</td>
<td>5.983</td>
<td>8.975</td>
<td>11.96</td>
<td>14.92</td>
<td>17.84</td>
<td>20.72</td>
<td>23.54</td>
</tr>
<tr>
<td>Capital dilution effect</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Saving effect</td>
<td>2.6918</td>
<td>1.7947</td>
<td>1.508</td>
<td>1.375</td>
<td>1.298</td>
<td>1.246</td>
<td>1.210</td>
<td>1.171</td>
<td>1.173</td>
</tr>
</tbody>
</table>

Notes: The capital dilution effect and saving effect are calculated according to (B.4) and (B.5), respectively, where \( x = 0.3, \delta = 0.05, g = 0.02, \) the population growth rate \( (n) \) and aggregate saving rate \( (s) \) in the current column refer to the initial equilibrium, and the corresponding entries in the next column refer to the final equilibrium. For example, 1/2.6918 refers to the capital dilution effect when \( n_1 = -0.06 \) and \( n_2 = -0.05, \) and is calculated according to (B.4). Note that the overall effect is positive (i.e., the saving effect dominates the capital dilution effect) if the product of the capital dilution and saving effects is larger than 1.
In each of the above cases, we conclude that the steady-state capital and the population growth rate are negatively related when the latter is in the range of \(-2\%\) to \(2\%\). With the empirical age-specific mortality rates of USA, the conclusion based on the baseline specification presented in Fig. 2 is robust to changes in parameter values of the model.
5. Quantitative assessment: with retirement

The previous section examines the relation between the population growth rate and capital accumulation in an overlapping-generations model with no retirement, similar to d’Albis (2007). Since the average retirement age in developed countries is substantially lower than the maximum age ($\Omega$) or even the life expectancy at birth, it is interesting to see whether the main conclusion in the previous section continues to hold or not when people retire. This section incorporates retirement into the d’Albis (2007) model. Specifically, we turn to the analysis with an exogenously imposed value of $T_r$ reflecting the actual experience of developed countries. It is assumed that $T_r = 45$, which corresponds to the retirement at actual age 65. Other aspects of the model are kept unchanged.

Fig. 6 shows that for parameters (including the retirement age) calibrated to the US economy, the negative relation between the population growth rate and steady-state capital per unit of effective labor is even sharper than that in Fig. 2. When the population growth rate increases anywhere in the range from −6% to 2%, steady-state capital per unit of effective labor decreases monotonically. The monotonic relation between population growth and steady-state capital in this version of the model contrasts with the non-monotonic relation between these two variables in the no-retirement version. The intuition of this difference, which can be traced to similar reasoning discussed in the previous section, is as follows. When retirement is anticipated, individuals save more in their working years, causing the aggregate saving rate to be higher for an economy with retirement, other things being equal. As a result, the saving effect caused by an increase in the population growth rate is more likely to be smaller and be dominated by the capital dilution effect, even when the population growth rate is very low initially.

In summary, with or without retirement, the relation between population growth rate and steady-state capital accumulation in the overlapping-generations model with US age-specific mortality rates is qualitatively the same as that in Diamond (1965) and Blanchard (1985).

6. Extension

As the main motivation of this paper is to assess quantitatively the effect of population growth on capital accumulation in the overlapping-generations framework used in d’Albis (2007), the model in earlier sections is kept as similar as possible. Nevertheless, the framework used in this paper can be extended to give insights to other issues. One possible extension is to incorporate a social security system in the retirement version of the model (with $T_r < \Omega$). The presence of the social security system is empirically relevant in many industrial countries; moreover, it may provide some justification of...
the mandatory retirement at age $T_r$ assumed in Section 5. To keep the model analytically tractable, we discuss below a simple social security system so that the changes from the model in Section 2 are minimal.

Consider a pure pay-as-you-go retirement system such that the social security payments by workers in any period are entirely transferred to the surviving retirees. Assume also that each retiree obtains a fraction $b$, the replacement ratio, of the current market wage, and each worker pays a fraction of his wage income. Under these assumptions and that of the stable population, it can be shown that the social security tax rate is time-invariant, and this tax rate, $\tau$, is determined according to

$$\tau \int_0^{T_r} e^{-nx(x)} dx = b \int_0^T e^{-nx(x)} dx. \quad (24)$$

In this environment, individuals’ budget constraints are changed, because of the changes in wage income (when working) and retirement income (after retiring). The new budget constraint of a cohort $s$ individual is given by

$$\frac{dZ(s, v)}{dv} = [\tau(v) + \mu(v - s)]Z(s, v) + [1 - (b, n)]w(v)N(s, v) + bw(v)[1 - N(s, v)] - C(s, v), \quad (25)$$

where the dependence of $\tau$ on $b$ and $n$ is written explicitly. Following similar steps as in Sections 2 and 3, it can be shown that the steady-state equilibrium of this economy is characterized by

$$(k^*)^2 - (1 - x)(k^*)^2 \left( \frac{\int_0^{T_r} e^{-x(r'-g)x} dx + b \int_0^T e^{-x(r'-g)x} dx}{\int_0^T e^{-x(r'-g)x} dx} \right)$$

$$(k^*)^2 \left( \frac{\int_0^{T_r} e^{-x(r+g-n)x} dx}{\int_0^T e^{-x(r)x} dx} \right) = (\delta + g + n)k^*. \quad (26)$$

Compared with the model without the social security system, a major difference is that a change in population growth rate ($n$) now has a direct effect on individual saving behavior, since a change in $n$ affects the individual’s budget constraint through the social security tax rate $\tau(b, n)$. Under the pure pay-as-you-go system modeled here, a higher population growth rate allows the retirees to be supported by more workers. With an unchanged replacement ratio, the social security tax rate will be reduced. As a result, one may expect that an individual will optimally choose a higher level of saving. Compared with the model in Section 5, changes in both individual saving behavior and age composition contribute to the saving effect in this economy with a pay-as-you-go system, and thus, the saving effect is more likely to offset the capital dilution effect. When the replacement ratio is 40%, which is close to the average figure at retirement for USA (Sheshinski, 2008, p. 2), Fig. 7 shows that the capital dilution effect dominates the saving effect for a smaller range (−1.2% to 2%) of population growth rates.20

One may also apply this model to study the effect of a change in the replacement ratio, $b$, on capital accumulation and related variables such as the interest rate. The model can further be extended to consider a social security system with both

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20 When $b = 40\%$, it can be shown that $\tau = 41.0\%$ when $n = -3\%$, and $\tau$ increases when $n$ decreases. A social security tax rate as high as 40% is not likely to be sustainable. Thus, Fig. 7 does not consider $n \leq -3\%$. 

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19 These specifications represent an idealized version of a pure pay-as-you-go system, and thus are unlikely to capture real-world systems perfectly. In the USA, for example, the social security benefit is adjusted for inflation but not for the change due to technological progress; the social security contribution is proportional to payroll earnings up to a specified level; and the Social Security system can own a Trust Fund, which is currently in surplus and is expected to be in deficit soon. Moreover, the survivors benefits and disability benefits of the system are not captured in our model. I am grateful to Ron Lee for his comments.
defined-benefit and defined-contribution elements, as in Abel (2003). In this extended model, a decrease in the replacement ratio, \( b \), in (24) may be interpreted as a policy change reducing the defined-benefit component relative to the defined-contribution component.

7. Concluding remarks

This paper examines the impact of fertility changes on capital accumulation in an overlapping-generations model with age-specific mortality rates. Motivated by the lack of robust evidence of population changes on asset prices (Poterba, 2001),\(^{21}\) d’Albis (2007) shows theoretically that the relation between fertility change and steady-state capital is not necessarily monotonically negative, since the capital dilution and saving effects are of opposite signs. Based on the empirical age-specific mortality rates of the USA, we find that while in principle the relation may not be monotonic, in practice the capital dilution effect usually dominates the saving effect. The results suggest that the negative relation between population growth and capital accumulation predicted in Diamond (1965) and Blanchard (1985) is, qualitatively, not misleading.

To understand the intuition of the quantitative results in this paper, we investigate the saving effect of a fertility change in the model used in d’Albis (2007). Since a fertility change does not affect an individual’s utility function and budget constraint directly, there is not much change in individual saving behavior.\(^{22}\) Thus, the saving effect is mainly caused by a change in age composition. Furthermore, it is found that with the age-specific mortality rates of the USA, the saving effect due to a change in age composition is in general not enough to offset the capital dilution effect. The study of this benchmark case of no direct effect (i.e., the demographic change does not affect individual saving behavior directly) is useful since the direct effect of a demographic change can usually be assessed a priori, but the indirect (general equilibrium) effect is not. Based on the analysis of this benchmark case, we further conjecture that for most situations, it is necessary for a demographic and/or policy change to increase individual saving rate in order to have a positive effect on capital accumulation. On the other hand, if such a change leads to a decrease in saving rate at the individual level, it is almost certain that the effect on capital accumulation is negative.

We also discuss how the model in Section 2 can be extended to incorporate a pay-as-you-go social security system. With appropriate modification, the framework used in Bommier and Lee (2003), d’Albis (2007) and this paper is helpful in analyzing the economic impact of various demographic and/or pension changes.

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Appendix A. Proof of Proposition 1

Define

\[
H(u) = \int_0^\Omega e^{-ux}(x) \, dx
\]  
(A.1)

and

\[
H_r(u) = \int_0^{T_r} e^{-ux}(x) \, dx
\]  
(A.2)

for \( u \in (-\infty, \infty) \). Based on (A.1) and (A.2), define

\[
J(u) = \frac{H_r(u - g)H(\sigma u + g + n + \rho)}{H(1 - \sigma u + \rho)H_r(n)},
\]  
(A.3)

---

21 The lack of unambiguous evidence (in terms of the sign and robustness) of population changes on economic variables has also been found by other researchers. For example, Kelley and Schmidt (1995) find that the effect of population growth rate on per capita output growth varies over time (insignificant in the 1960s and 1970s, but significant and negative in the 1980s) and varies with the level of development (negative in developing countries, but sometimes positive in industrial countries). An and Jeon (2006) find that for OECD countries in the post World War Two era, economic growth rates rise initially and then fall with population aging.

22 Note that if a pay-as-you-go social security system is present, then a change in fertility rate can still have a direct effect on an individual’s budget constraint (see (24) and (25)), even if there is purely life-cycle saving motive.
Thus, the l'Hôpital's rule to conclude that
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{H(u)}{H(u)} = \frac{\int_0^x x e^{-ax} dx}{\int_0^x e^{-ax} dx} = \frac{h(u)}{H(u)}
\]
and
\[
\lim_{x \to 0^+} H(u) = \lim_{x \to 0^+} \frac{H(u)}{H(u)} = \frac{\int_0^x x e^{-ax} dx}{\int_0^x e^{-ax} dx} = \frac{f'(u)}{H(u)}
\]
where \(H'(\cdot)\) is the derivative of \(H(\cdot)\) and \(H'_*(\cdot)\) is the derivative of \(H_*(\cdot)\). Note that \(H(u)\) and \(H'_*(u)\) are positive and decreasing in \(u\), since \(f(x)\) is positive for \(x \in (0, \Omega)\). As a result, \(f(u)\) is also positive.

Based on (15), (16), (18), (A.1)–(A.3), the equilibrium condition (22) can be rewritten as
\[
\phi(k') = k',
\]
where function \(\phi(k)\) is defined as
\[
\phi(k) = \frac{(1 - x)k^2}{2k^{x-1} - (\delta + g + n)}[I[(x) - 1] - 1].
\]

A.1. Existence

It turns out to be easier to prove existence of the steady-state equilibrium when one focuses on \(r\) instead of \(k\), where \(r\) and \(k\) are related according to \(r = xk^{x-1} - \delta\). It is straightforward to show that \(\phi(k') = k'\) in (A.6) is equivalent to
\[
\dot{z}(r^*) = 0,
\]
where
\[
\dot{z}(r) = (1 - z) \frac{[I(r) - 1]}{r - (g + n)} - \frac{z}{r + \delta}.
\]
When \(r\) tends to \(-\delta\) from above, \(z/(r + \delta)\) tends to \(+\infty\), and \([I(r) - 1]/[r - (g + n)]\) tends to \([I(-\delta) - 1]/(\delta + g + n)\), which is finite.\(^{24}\) Thus,
\[
\lim_{r \to -\delta} \dot{z}(r) = -\infty.
\]
Next, consider the limit of \(\dot{z}(r)\) when \(r\) tends to \(+\infty\). First, following the proof similar to that for Lemma 8 in d’Albis (2007), it can be shown that when condition (23) is satisfied, \(\lim_{u \to \infty} f(u) = +\infty\) and \(\lim_{u \to \infty} f'(u) = +\infty\), where \(f(\cdot)\) is the derivative of \(f(\cdot)\).

Using the l'Hôpital’s rule, it can be shown that
\[
\lim_{r \to +\infty} \dot{z}(r) = (1 - z) \lim_{r \to +\infty} \frac{[I(r) - 1]}{r - (g + n)} = (1 - z) \lim_{r \to +\infty} \frac{f(r)}{r - (g + n)} = +\infty.
\]

Since \(\dot{z}(r)\) is continuous for \(-\delta < r < +\infty\), one concludes from (A.10) and (A.11) that \(\dot{z}(r^*) = 0\) exists for \(-\delta < r^* < +\infty\). Equivalently, \(\phi(k') = k'\) exists for \(0 < k^* < +\infty\).

A.2. Uniqueness

(This part of the proof follows Proposition 3 of d’Albis, 2007 closely, except that the Cobb–Douglas production function is used in this paper.) One can prove uniqueness of the steady-state equilibrium \(k^*\) by showing that \(\phi'(k^*) < 0\). When (23) is

\(^{23}\) The original problem is to look for \(\phi(k') = k'\) where \(0 < k' < +\infty\). After transformation, the problem is to look for \(\dot{z}(r^*) = 0\) where \(-\delta < r^* < +\infty\). I am grateful to a referee for this suggestion.

\(^{24}\) It is implicitly assumed in d’Albis (2007) and this paper that \(\dot{z} + g + n\) is strictly positive. This restriction is likely to be satisfied for most countries that economists are interested in.

\(^{25}\) Since both the numerator and denominator of \([I(r) - 1]/[r - (g + n)]\) are continuous in \(-\delta < r < +\infty\), discontinuity of this term could occur only at \(r = g + n\). However, it can be shown that this term is finite at \(r = g + n\), as follows. First, it is easy to show from (A.3) that \(f(g + n) = 1\). Second, one applies the l'Hôpital’s rule to conclude that
\[
\lim_{r \to g + n} \frac{[I(r) - 1]}{r - (g + n)} = \lim_{r \to g + n} \frac{f(r)}{r - (g + n)} = f(g + n).
\]

Third, from (A.3), the first derivative of \(f(u)\) is given by
\[
f'(u) = \sigma h(-au + g + n + \sigma \rho) + (1 - \sigma) h((1 - \sigma)u + \sigma \rho) - h_t(u - g)\]
Thus, \(f(g + n) = h((1 - \sigma)(g + n) + \sigma \rho) - h_t(n)\), which is finite.
satisfied, a sufficient condition for $\phi'(k') < 0$ is
\[
\frac{d}{dk} \left[ \frac{k}{f(k) - kf'(k)} \right] \geq 0.
\] (A.12)

For the Cobb–Douglas production function (13), one obtains
\[
\frac{d}{dk} \left[ \frac{k}{f(k) - kf'(k)} \right] = \frac{d}{dk} \left( \frac{k^{1-z}}{1 - z} \right) = k^{-2} > 0.
\]

Therefore, (A.12) holds. This completes the proof.

Appendix B. Capital dilution and saving effects

This appendix provides a decomposition of the effect of a change in the population growth rate on capital accumulation. As illustrated in Fig. A1, the initial equilibrium is represented by point A, the intersection of the actual investment curve $s_1k^\alpha$ and the break-even investment line $(\delta + g + n_1)k$, where $n_1$ is the initial population growth rate and $s_1$ is the associated aggregate saving rate. After the population growth rate increases from $n_1$ to $n_2$ (and aggregate saving rate changes endogenously from $s_1$ to $s_2$), the new equilibrium is given by point C.\(^{26}\)

With the Cobb–Douglas production function, it is easy to show that at equilibrium A, $s_1(k_A^\alpha) = (\delta + g + n_1)k_A^\alpha$ and thus,
\[
k_A^\alpha = \left( \frac{s_1}{\delta + g + n_1} \right)^{1/(1-\alpha)}.
\] (B.1)

Similarly, at equilibrium C, we have
\[
k_C^\alpha = \left( \frac{s_2}{\delta + g + n_2} \right)^{1/(1-\alpha)}.
\] (B.2)

To decompose the change in $k^\alpha$ from points A to C into the capital dilution and saving effects, we first consider the effect of a change in the population growth rate when aggregate saving rate is assumed to remain hypothetically at $s_1$. The effect is represented as a change from points A to B (where $k = k_B$) in Fig. A1. Using the same formula as before, we have
\[
k_B = \left( \frac{s_1}{\delta + g + n_1} \right)^{1/(1-\alpha)}.
\] (B.3)

Combining (B.1) and (B.3), we conclude that\(^{27}\)
\[
k_B/k_A^\alpha = \left( \frac{\delta + g + n_1}{\delta + g + n_2} \right)^{1/(1-\alpha)}.
\] (B.4)

According to (B.4), an increase in the population growth rate (i.e., $n_2 > n_1$) always leads to a decrease in capital per unit of effective labor. This is the capital dilution effect.

\(^{26}\) Fig. A1 is based on $n_1 = 1\%$ and $n_2 = 2\%$ (see Table 1 for the values of other parameters). Point C would have been on the right of point A if $n_1 = -6\%$ and $n_2 = -5\%$.

\(^{27}\) One nice feature of this decomposition is that the ratio in (B.4) is independent of the aggregate saving rate.
The saving effect is caused by the endogenous change in aggregate saving rate (when the population growth rate remains at the new level, \( n_2 \)). Graphically, this is represented as a change from points B to C in Fig. A1. Thus, combining (B.2) and (B.3), the saving effect can be represented by the ratio

\[
\frac{k_C}{k_B} = \left( \frac{s_2}{s_1} \right)^{1/(1-\delta)}.
\]

The decomposition for the model with US age-specific mortality rates, based on (B.4) and (B.5), is given in Table 1.

References