Price Setting with Interdependent Values

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Abstract
We consider a take-it-or-leave-it price offer game under value interdependence. The main result is a simple but general sufficient condition that ensures existence and uniqueness of a separating equilibrium. The condition amounts to the monotonicity of what we call the Myerson virtual profit. Prices are shown to be higher than under signal revelation, thus reflecting a signalling premium.

Keywords: Adverse selection, lemons, bilateral bargaining, two-sided private information

JEL Codes: D82, D83

1 Introduction
Consider a setting where a seller, who has a single unit for sale, offers a price to a buyer who has a unit demand. When the seller has some information relevant to the buyer, then adverse selection may impede trade even as the seller’s information is transmitted to the buyer through price. An important question then is under what conditions a separating equilibrium, where the seller’s information is fully revealed to the buyer, exists in such a setting.

Jullien and Mariotti (2006) have recently looked at an example of such a model with linear valuations, and have shown that a separating equilibrium exists. Cai, Riley, and Ye (2007) have

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obtained a simple, but powerful and general sufficient condition for existence of a separating equilibrium. Their condition effectively amounts to the monotonicity of the marginal profit à la Myerson (1981) and Bulow and Roberts (1989).\(^\text{1}\)

Both Jullien and Mariotti (2006) and Cai, Riley, and Ye (2007) restrict attention to the case where the seller’s information is independent from the buyer’s.\(^\text{2}\) In this note, we generalize their condition to an environment where the buyer’s and seller’s information is correlated, and the values are interdependent.

There are many real-life examples where information is multi-dimensional and such an extension would be logical.\(^\text{3}\) In addition, there are many examples (e.g. real estate and natural resource sales) where the buyer’s information is relevant to the seller. As this gives rise to the “winner’s curse” for the seller in addition to adverse selection for the buyer, this setting presents additional challenges and requires a careful analysis. In fact, our methods are different from the ones in the cited papers and are closer in spirit to the geometric methods that have been used in the double auction literature, in particular in Satterthwaite and Williams (1989), and, more recently, in Kadan (2007).

In our model, the values are in general interdependent. We assume that the values are monotone in the buyer and seller signals, but do not impose any additional conditions such as single-crossing.\(^\text{4}\)

The condition for the existence of a separating equilibrium in our paper is essentially the same as in Cai, Riley, and Ye (2007). That is, the seller’s marginal profit is required to be monotone in the buyer’s signal. The only difference is that in our model, the seller’s marginal profit also depends on the buyer’s signal through its impact on the seller’s valuation. Moreover, the equilibrium pricing strategy is shown to be unique in the class of strictly increasing strategies, without the need to impose any further refinements.\(^\text{5}\)

The price offered in the separating equilibrium is shown to always involve a counter-efficient

\(\text{Footnotes}\)

1See also Lamy (2010) for a corrected version of their condition in the multi-buyer case.
2In addition, they consider a setting with several buyers.
3See the discussion in the concluding section of Cai, Riley, and Ye (2007).
4In addition, we assume differentiability and the standard MLRP condition on the conditional distribution of the buyer information.
5Mailath (1987) considers a general signalling problem and imposes a set of sufficient conditions for existence and uniqueness of a separating equilibrium. His setting does not cover two-sided private information as in our paper.
signalling premium, making it above the price that would be offered if the seller’s signal were revealed to the buyer.\textsuperscript{6} For intuition, consider a “lemons” environment, which is included as a special case. When the seller’s information is private, the seller has an additional incentive to mimic the one with a higher signal because doing so would convince the buyer that the quality is higher, thereby increasing the probability of acceptance. To countervail this incentive, it is necessary to reduce the seller’s acceptance probabilities relative to the full revelation level. This implies a higher price. Here, the signalling premium is shown to exist as a general phenomenon.

2 Model

There is a buyer and a seller. The seller has a unit of a good to sell, and the buyer has a unit demand. We consider a price-setting game: the seller makes a take-it-or-leave-it offer to the buyer, who may either accept or reject it.

Before the game begins, the buyer and the seller receive signals $x_B$ and $x_S$ that affect their ex-post valuations of the good, $u_B(x_B, x_S)$ and $u_S(x_S, x_B)$. The signals $x_B$ and $x_S$ will be sometimes referred to as the buyer’s and seller’s types. The buyer and the seller do not observe the signal received by the opposite side, only their own signals.

We allow $x_B$ to be correlated with $x_S$. The signals are drawn prior to the game from a joint distribution supported on $[0, 1]^2$. We will need the following regularity assumption on the conditional distribution of the buyer’s signal, $F_B(x_B|x_S)$.

**Assumption 1** (Regularity of density). The distribution $F_B(x_B|x_S)$ has density $f_B(x_B|x_S)$, continuously differentiable and positive on $[0, 1]^2$.

To capture the idea that signals are positively associated, we impose the Monotone Likelihood Ratio Property (MLRP).

**Assumption 2** (MLRP). For any $x_B \leq \hat{x}_B, x_S \leq \hat{x}_S$, we have

$$\frac{f_B(\hat{x}_B|x_S)}{f_B(x_B|x_S)} \leq \frac{f_B(\hat{x}_B|\hat{x}_S)}{f_B(x_B|\hat{x}_S)}.$$

We assume that the buyer’s and seller’s valuations are increasing in the signals, and are also regular in the sense defined below.

\textsuperscript{6}The existence of a signalling premium also follows from the analysis in Jullien and Mariotti (2006). The second part of their Proposition 1 shows that the probability of trade is reduced under private information.
Assumption 3 (Monotonicity and differentiability of valuations). The buyer’s and seller’s valuations $u_B(x_B, x_S)$ and $u_S(x_S, x_B)$ are twice continuously differentiable, strictly increasing in own signals and nondecreasing in the partner signals, with

$$\frac{\partial u_B(x_B, x_S)}{\partial x_B} > 0, \quad \frac{\partial u_B(x_B, x_S)}{\partial x_S} \geq 0,$$

$$\frac{\partial u_S(x_S, x_B)}{\partial x_B} > 0, \quad \frac{\partial u_S(x_S, x_B)}{\partial x_S} \geq 0.$$

Assumption 3 rules out a pure “lemons” environment because the buyer is assumed to have its own signal affecting his valuation. In our model, in order to sustain a separating equilibrium in pure strategies, it is necessary that the buyer would reject higher offers with a higher probability. Our assumption that the buyer’s utility is strictly increasing in own signal will ensure that the buyer will reject some offers not only because the quality is low, but also because his own signal is low, which provides a degree of horizontal differentiation on the buyer side.

2.1 Benchmark: Equilibrium under revelation

As a benchmark, consider a model with one-sided private information: the seller’s signal $x_S$ is observable to the buyer. We will refer to this setting as revelation. Then the buyer’s responding strategy is obtained as a minimal signal $x_B^*(p, x_S)$ such that the seller’s offer $p$ is acceptable to a buyer,

$$x_B^*(p, x_S) = \inf\{x_B \in [0, 1] : u_B(x_B, x_S) \geq p\}$$

Note that the offer $p$ can be so low that it is acceptable to a buyer with any signal if $p \leq u_B(0, x_S)$; in this case, $x_B^* = 0$. Or, it can be so high that it is never acceptable and then $x_B^*$ is not defined. For any price $p \in [u_B(0, x_S), u_B(1, x_S)]$, the buyer’s reservation cutoff $x_B^*(p, x_S)$ is determined from

$$u_B(x_B^*(p, x_S), x_S) = p. \tag{1}$$

For $p < u_B(0, x_S)$, the reservation cutoff is $x_B^*(p, x_S) = 0$, while for $p > u_B(1, x_S)$, it is given by $x_B^*(p, x_S) = 1$. Given the buyer’s responding strategy $x_B^*(p, x_S)$, the seller’s problem is to choose price $p$ to maximize the expected profit.
It turns out to be more convenient to reformulate the problem in terms of \( q \), the induced probability of sale.\(^7\) If the marginal buyer type is \( x_B \), then

\[
q = 1 - F_B(x_B|x_S) \implies x_B(q) = F_B^{-1}(1 - q|x_S)
\]

and the expected seller’s profit as a function of \( q \) is

\[
\pi_S(q, x_s) = q u_B(x_B(q), x_s) - \int_{x_B(q)}^{1} u_S(x_S, \bar{x}_B)f_B(\bar{x}_B|x_S)d\bar{x}_B,
\]

so the marginal profit is

\[
\frac{\partial \pi_S(q, x_s)}{\partial q} = u_B(x_B(q), x_s) + q \frac{\partial u_B(x_B(q), x_s)}{\partial x_B}x_B(q) - u_S(x_S, x_B(q))x_B'(q)f_B(x_B(q)|x_S).
\]

As

\[
x_B(q) = F_B^{-1}(1 - q|x_S) \implies x_B'(q) = -\frac{1}{f_B(x_B(q)|x_S)},
\]

after some algebra, the marginal profit simplifies to

\[
\frac{\partial \pi_S(q, x_S)}{\partial q} = J_B(x_B(q), x_S) - u_S(x_S, x_B(q))
\]

\[
\equiv J(x_B(q), x_S),
\]

(2)

(3)

where \( J_B(x_B, x_S) \) is the Myerson virtual value,

\[
J_B(x_B, x_S) \equiv u_B(x_B, x_S) - \frac{\partial u_B(x_B, x_S)}{\partial x_B} \cdot \frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)},
\]

(4)

and \( J(x_B, x_S) \) is what we shall call the seller’s virtual profit.

Central to our analysis is the following standard monotonicity assumption on the virtual profit function.

**Assumption 4** (Monotonicity of virtual profit). For all \( x_B, x_S \in [0,1] \), we have

\[
\frac{\partial J(x_B, x_S)}{\partial x_B} > 0.
\]

The iso-marginal profit curve, or the J-curve for short,

\[
\{(x_B, x_S) \in [0,1]^2 : J(x_B, x_S) = 0\},
\]

\(^7\)This approach is motivated by Bulow and Roberts (1989).
separates the square \([0,1]^2\) that represents the set of all possible buyer and seller signals, into two parts. The marginal profit is positive below the J-curve, and negative above it.

Let \(x_B^*(x_S) \in [0,1]\) be the minimal buyer type for whom the \(x_S\)-seller’s optimal offer is acceptable. The optimal price offer is equal to \(u_B(x_B^*(x_S), x_S)\). Since \(J(x_B, x_S)\) is increasing in \(x_B\), the marginal profit is decreasing in \(q\), and there are three possibilities for \(x_B^*(x_S)\).

- If \(J(0, x_S) > 0\), then the marginal profit is positive for all price offers that can ever be accepted, and therefore the optimally chosen price is the highest one acceptable to all buyers. This implies that \(x_B^*(x_S) = 0\).
- If \(J(1, x_S) < 0\), then the marginal profit is negative for all price offers, and there is no trade. In this case, we set \(x_B^*(x_S) = 1\) by convention.
- If \(J(0, x_S) \leq 0\) and \(J(1, x_S) \geq 0\), then \(x_B^*(x_S)\) is uniquely determined by \(J(x_B^*(x_S), x_S) = 0\), i.e. lies on the J-curve.

From now on, we assume that all seller types trade with a positive probability.

**Assumption 5.** We have

\[
x_B^*(x_S) < 1 \quad \forall x_S \in [0,1].
\]

**Remark 1.** More primitive conditions that ensure the monotonicity of the virtual profit \(J\) can be provided. First, notice that if the seller’s valuation \(u_S(x_S, x_B)\) does not depend on \(x_B\), then it is sufficient to require that the virtual value \(J_B(x_B, x_S)\) is increasing in \(x_B\) and has a positive slope, \(\partial J_B(x_B, x_S)/\partial x_B > 0\). This generalizes the condition obtained in Cai, Riley, and Ye (2007) for auctions by allowing the seller’s and the buyer’s signals to be correlated. Second, if \(u_S(x_S, x_B)\) does depend on \(x_B\) so that we have genuine value interdependence, then the monotonicity of \(J\) in \(x_B\) is implied by the following more primitive conditions, most of which are standard in the literature.

1. **Single crossing:** The difference \(u_B(x_B, x_S) - u_S(x_S, x_B)\) is increasing in \(x_B\).

2. The buyer’s utility function \(u_B(x_B, x_S)\) is convex in \(x_B\) for any \(x_S \in [0,1]\), so that \(\frac{\partial u_B(x_B, x_S)}{\partial x_B}\) is nondecreasing in \(x_B\).
3. The conditional distribution of the buyer’s signal, \( F_B(x_B|x_S) \) has nondecreasing hazard rate, so that \( \frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)} \) is a nonincreasing function of \( x_B \).

It is not hard to provide specific examples of utility functions satisfying (1) and (2). For the condition on the signals (3) above, one needs to ensure that it is compatible with the MLRP, which is the maintained assumption. One such example is the FGM family of densities, introduced to model affiliation in Kosmopoulou and Williams (1998), and applied more recently to double auctions in Kadan (2007) and Gresik (2011). This family is defined as

\[
f(x_B, x_S) = 1 + \kappa (1 - 2x_S)(1 - 2x_B) \quad (\kappa \geq 0),
\]

and it can be easily verified that it satisfies the MLRP. Moreover, since the marginals are uniform \([0, 1]\) distributions, the conditional density \( f(x_B|x_S) = f(x_B, x_S) \), and it is log-concave in \( x_B \) for any fixed \( x_S \in [0, 1] \). Since

\[
\frac{f_B(x_B|x_S)}{1 - F_B(x_B|x_S)} = - \frac{d \log (1 - F_B(x_B|x_S))}{dx_B},
\]

this implies that FGM distributions \( F(x_B|x_S) \) have hazard rate increasing in \( x_B \).

### 2.2 Separating equilibrium with private signal

In this section, we prove our main result: even when the seller’s signal is unobservable to the buyer, there is a unique separating equilibrium in monotone strategies. That is, in equilibrium the seller fully reveals its signal to the buyer. Moreover, we show that the price will involve a signalling premium.

We first show that, in parallel to the revelation setting, the buyer’s best-response responding strategy is still characterized by a cutoff rule.

**Lemma 1.** For any (measurable) seller’s strategy \( S : [0, 1] \to \mathbb{R}_+ \), the buyer’s best-response strategy to a price offer \( p = S(x_S) \) for some \( x_S \in [0, 1] \) is characterized by a cutoff \( X_B(p) \) such that the buyer accepts the offer if and only if \( x_B \geq X_B(p) \).

**Proof.** The buyer’s expected profit upon accepting a price offer \( p \) is

\[
\Pi_B(x_B, p) = \int_{\{\tilde{x}_S : S(\tilde{x}_S) = p\}} (u_B(x_B, \tilde{x}_S) - p) dF_S(\tilde{x}_S|x_B), \tag{5}
\]

\(^8\text{As before, we assume that whenever the buyer is indifferent between accepting or not, he will accept.}\)
and, because the integrand is increasing in $x_B$, $\Pi_B(x_B, p)$ is continuous and increasing in $x_B$ under first-order stochastic dominance implied by the MLRP assumption (Assumption 2). So if $p$ is acceptable to a buyer with signal $x_B$, $\Pi_B(x_B, p) \geq 0$, it is also acceptable for a buyer with a higher signal $x'_B > x_B$, and it follows that the buyer’s strategy can be defined as the lowest signal $X_B(p)$ such that the price $p$ is acceptable, $X_B(p) = \inf\{x \in [0, 1] : \Pi_B(x, p) \geq 0\}$. 

In this paper, we investigate fully separating equilibria (S-equilibria), i.e. those equilibria where (i) there is a unique seller type $X_S(p)$ offering price $p$, and (ii) there is a unique buyer type $X_B(p)$ such that this type and all types above it find the price offer $p$ acceptable. Moreover, we restrict attention to equilibria in continuous and monotone strategies.

**Definition 1** (S-equilibrium). An S-equilibrium is defined as a perfect Bayesian equilibrium such that the seller’s strategy $S(\cdot)$ is a continuous and increasing function.

This assumption implies that the seller’s equilibrium strategy is fully type-revealing, and the inverse strategy

$$X_S(p) = \inf\{x \in [0, 1] : S(x) \geq p\}$$

is nondecreasing and continuous. We can now define the range of prices that will be offered, $[\underline{p}, \bar{p}]$, where

$$\underline{p} = S(0), \quad \bar{p} = S(1).$$

A pair of equilibrium strategies is shown in Figure 1. Since $S(\cdot)$ is assumed to be increasing, it is clear that in any Perfect Bayesian equilibrium, the buyer, upon receipt of the offer $p = S(x_S)$, will infer the seller’s type $x_S$ and respond accordingly. That is, the offer will be accepted if $u_B(x_B, x_S) \geq p$, and rejected otherwise. So for $p \in [\underline{p}, \bar{p}]$, the equilibrium responding strategy is given by

$$X_B(p) = \inf\{x \in [0, 1] : u_B(x, X_S(p)) \geq p\}.$$  

In a separating equilibrium, the best-response by the buyer is to accept an offer $p$ if $u_B(x_B, X_S(p)) \geq p$ and reject otherwise, so the best-response strategy $X_B(p)$ for all $p \in [\underline{p}, \bar{p}]$ is given as a (unique) solution to

$$u_B(X_B(p), X_S(p)) = p. \quad (6)$$
In equilibrium, the seller with signal $x_S$ will choose $p$ optimally, maximizing the expected profit function

$$\Pi_S(x_S, p) = \int_{X_B(p)}^1 [p - u_S(x_S, \tilde{x}_B)] f_B(\tilde{x}_B|x_S)d\tilde{x}_B,$$

resulting in a F.O.C. at all points of differentiability.\(^9\)

$$\frac{\partial \Pi_S(x_S, p)}{\partial p} = -X_B'(p)(p - u_S(x_S, X_B(p))) f_B(X_B(p)|x_S) + 1 - F_B(X_B(p)|x_S) = 0.$$ 

Using (6) and simplifying, we obtain the differential equation for the buyer’s inverse strategy $X_B(p)$:\(^{10}\)

$$X_B'(p) = \frac{1 - F_B(X_B(p)|X_S(p))}{f_B(X_B(p)|X_S(p))} \frac{1}{u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p))}. \quad (7)$$

It is convenient to totally differentiate (6),

$$\frac{\partial u_B}{\partial x_B} X_B'(p) + \frac{\partial u_B}{\partial x_S} X_S'(p) = 1,$$

\(^9\)A monotone function is differentiable almost everywhere.

\(^{10}\)In line with usual notation for differential equations, in the r.h.s. we suppress the dependence of $x_B$ and $x_S$ on $p$. 

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Figure 1: A pair of equilibrium strategies.
and then substitute the slope $X'_S(p)$ from the seller’s F.O.C. (7), yielding after some manipulations

\[
X'_S(p) = \frac{1}{\frac{\partial u_B}{\partial x_S} \left( 1 - \frac{\partial u_B}{\partial x_B} X'_B(p) \right)} \left(J(X_B(p), X_S(p)) - u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p))\right)
\]  

Equations (7) and (8) were derived for all points of differentiability of $X_B(\cdot)$ and $X_S(\cdot)$. However, as the following lemma shows, both $X_B(\cdot)$ and $X_S(\cdot)$ must in fact be continuously differentiable.

**Lemma 2.** The seller’s inverse offer strategy $X_S(\cdot)$ and the buyer’s responding strategy $X_B(\cdot)$ are continuously differentiable functions.

**Proof.** Since $S(\cdot)$ is a strictly increasing, continuous function according to the definition of an S-equilibrium, its inverse $X_S(\cdot)$ is also strictly increasing and continuous. We now show that the definition of the buyer’s responding strategy as the function $X_B(p)$ that solves (6) implies that $X_B(\cdot)$ is likewise continuous. We argue by contradiction. If $p_0$ is a point of discontinuity of $X_B(p)$, then there exist two sequences of prices $\{p_n\}, \{p'_n\}$ such that $p_n, p'_n \to p_0$ and $X_B(p_n) \to x^1_B, X_B(p'_n) \to x^2_B$ where $x^1_B \neq x^2_B$. Without loss of generality, assume $x^2_B > x^1_B$. Since $u_B(\cdot, \cdot)$ is also continuous (indeed, continuously differentiable), passing to the limit in (6) along the sequences $p_n$ and $p'_n$ yields

\[
u_B(x^1_B, X_S(p_0)) = p_0 = u_B(x^2_B, X_S(p_0)),
\]

but this contradicts our Assumption 3 according to which the buyer’s valuation is strictly increasing in $x_B$. So $X_B(\cdot)$ must be continuous. As a monotone function, $X_S(\cdot)$ is differentiable almost everywhere. Then, equation (6) implies that $X_B(\cdot)$ is also differentiable almost everywhere. Now all the functions appearing on the r.h.s. of (7) and (8), namely $u_B(\cdot, \cdot), u_S(\cdot, \cdot)$ and $F_B(\cdot | \cdot), f_B(\cdot | \cdot)$, are continuous by assumption. Also, $X_S(p)$ is continuous by definition of an S-equilibrium, and we have just shown that $X_B(\cdot)$ is also continuous. It follows that the r.h.s. of (7) and (8) are continuous functions of $p$, and therefore the derivatives $X'_B(p)$ and $X'_S(p)$ appearing in the l.h.s. can be extended by continuity to the entire interval $[p, \overline{p}]$. There-
Figure 2: A phase portrait of an equilibrium. The equilibrium curve is marked by $E$. The monotonicity region corresponds to the part of the square below the $J(x_B, x_S) = 0$ curve, marked as $J$. The ex-post efficiency region lies below the $u_B(x_B, x_S) = u_S(x_S, x_B)$ curve.

Therefore in any S-equilibrium, both $X_B(\cdot)$ and $X_S(\cdot)$ are continuously differentiable, not merely continuous.

So far we have shown that the first-order conditions given by the system of differential equations (7) and (8) are necessary for S-equilibrium. In order to complete the characterization of an equilibrium candidate, we need to pin down the initial condition for this system. This is done in the following lemma, which also establishes uniqueness of the equilibrium candidate.

**Lemma 3.** For any S-equilibrium, $S(0) = u_B(\bar{x}_B, 0)$, where $\bar{x}_B \equiv x_B^*(0)$ is the point of intersection of the J-curve with the horizontal axis, or $\bar{x}_B = 0$ if $J(x_B, 0) > 0 \forall x_B \in [0,1]$.

**Proof.** Equivalently, we need to show that

$$X_B(p) = \bar{x}_B, \quad X_S(p) = 0,$$

where $p = u_B(\bar{x}_B, 0)$.

Refer to Figure 2. In this figure, the J-curve defined in the previous section separates the feasible signal space $[0,1]^2$ into the upper and lower region. Note that the solution cannot
enter the upper region since monotonicity fails there, either $X'_S(p) < 0$ if $u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p)) > 0$, or $X'_B(p) < 0$ if $u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p)) < 0$. So the initial condition for the buyers must be $X_B(p) \geq x_B$.

We now show that any $X_B(p) > x_B$ will give the seller the incentive to deviate to a lower price. For an out-of-equilibrium offer $p < \underline{p}$, buyer’s beliefs are not pinned down by the Bayes rule. Still, in a Perfect Bayesian equilibrium, the buyer will best-respond given some beliefs about the seller’s types $x_S$. So we consider the worst-case scenario when the buyer believes that this offer came from this seller type $x_S = 0$, which is in fact the case. This belief corresponds, for $p < \underline{p}$, to the highest buyer’s marginal type $X_B(p)$ possible, defined as the unique solution to $u_B(X_B(p), 0) = p$, since with these beliefs, the buyer with any signal $x_B$ assigns the lowest possible value to the object, $u_B(x_B, 0)$. So the seller’s profit from such a deviation is bounded from below by $\Pi_S(X_B(p))$, where

$$\Pi_S(x_B) \equiv \int_{x_B}^{1} (u_B(x_B, 0) - u_S(0, \tilde{x}_B)) f_B(\tilde{x}_B|0) d\tilde{x}_B.$$ 

After some algebra, the slope of this profit function at $x_B = X_B(p)$ can be shown to be

$$\Pi'_S(X_B(p)) = -J(X_B(p), 0) \cdot f_B(X_B(p)|0).$$ 

If $X_B(p) > x_B$, then $J(X_B(p), 0) > J(x^*_B, 0) \geq 0$ and this slope is negative. This means that even under the most unfavourable buyer beliefs, and therefore for any beliefs, the seller’s expected profit can be increased by offering a lower price $p < \underline{p}$. This is a contradiction. So the only remaining possibility is $X_B(p) = x_B$. $\square$

Let

$$\mathcal{M} \equiv \{(x_B, x_S) \in [0, 1]^2 : J(x_B, x_S) \geq 0\}$$ 

be the domain where the equilibrium candidate is monotone, $X'_S(p), X'_B(p) > 0$. The following lemma shows that the r.h.s. of the differential equations (7) and (8) define a vector field on $\mathcal{M}$.

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Lemma 4 (Continuously differentiable vector field). The mapping

\[
(x_B, x_S) \mapsto \left( \frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)} u_B(x_B, x_S) - u_S(x_S, x_B), \frac{1}{\partial u_B(x_B, x_S)} \frac{J(x_B, x_S)}{u_B(x_B, x_S) - u_S(x_S, x_B)} \right)
\]

defines a continuously differentiable vector field on \(\mathcal{M}\).

Proof. Assumption 3 implies that \(u_B, u_S\) and \(\partial u_B/\partial x_B, \partial u_B/\partial x_S\) are continuously differentiable. It only remains to verify that the difference \(u_B(x_B, x_S) - u_S(x_S, x_B)\) is bounded from below on \(\mathcal{M}\) by a positive constant. The mean-value theorem applied to \(J(x_B, x_S)\) on \(\mathcal{M}\) implies for some \(\tilde{x}_B \in [x_B^*(x_S), 1]\):

\[
J(x_B, x_S) = J(x_B^*(x_S), x_B) + \frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B} (\tilde{x}_B - x_B^*(x_S))
\geq 0 + \frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B} (x_B - x_B^*(x_S)).
\]

The definition of \(J\) then implies

\[
u_B(x_B, x_S) - u_S(x_S, x_B) \geq \frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)} \frac{\partial u_B}{\partial x_B} + \frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B} (x_B - x_B^*(x_S)).
\]

Assumption 4 implies that the derivative of \(J\) with respect to \(x_B\) is bounded from below by a positive constant,

\[
\frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B} \geq a > 0.
\]

Also, the boundedness of \(f_B(x_B, x_S)\) from both above and below on \([0, 1]^2\) implies

\[
\frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)} \geq \frac{f_B}{\int_B} (1 - x_B).
\]

while Assumption 3 implies \(\partial u_B/\partial x_B\) is bounded from below on \([0, 1]^2\) by a positive constant, say \(\phi > 0\). Therefore,

\[
u_B(x_B, x_S) - u_S(x_S, x_B) \geq \frac{\int_B}{\int_B} \phi (1 - x_B) + a (x_B - x_B^*(x_S)) \geq \min\{\frac{\int_B}{\int_B} \phi, a\} \min\{1 - x_B, x_B - x_B^*(x_S)\} \geq \min\{\frac{\int_B}{\int_B} \phi, a\} \frac{1 + x_B^*}{2} > 0.
\]

So the r.h.s. of (7) and (8) in fact define a continuously differentiable vector field on \(\mathcal{M}\). \(\square\)
Our main result is the following proposition that shows existence and uniqueness of a Perfect Bayesian S-equilibrium.

**Proposition 1.** There exists a unique perfect Bayesian S-equilibrium, characterized by the pair of differential equations (7) and (8), with the initial conditions (9). For price offers outside the equilibrium range \([p, \bar{p}]\), this equilibrium is supported by out-of-equilibrium buyer beliefs as follows:

- For \(p < \underline{p}\), then the buyer believes that the offer originated from the seller with the lowest signal, \(x_S = 0\);
- For \(p > \bar{p}\), the buyer’s beliefs are unrestricted.

**Proof.** Step 1: The solution to the system of differential system exists, is unique and monotone, and defines an equilibrium candidate.

By Lemma 4, the right-hand sides of (7) and (8) define a vector field, continuously differentiable on \(\mathcal{M}\). Since the initial condition for the system (7) and (8), \((x_B, 0)\), is at the boundary of \(\mathcal{M}\), a fundamental result in the theory of differential equations then implies that there is a unique integral curve passing through \((x_B, 0)\). This curve is entirely contained in \(\mathcal{M}\), and therefore the corresponding \(X_B(\cdot)\) and \(X_S(\cdot)\) functions are monotone with \(X_B'(p), X_S'(p) > 0\). To claim that this integral curve defines a (unique) S-equilibrium candidate, we need to verify that it exits the square through the upper edge \([0, 1] \times \{1\}\). Equivalently, this will show that the solution can be extended to the entire interval \([\underline{p}, \bar{p}]\), where \(\bar{p} = u_B(x_B, 1)\) for some \(x_B \in [x_B^*(1), 1]\).

The argument is geometric. Refer to Figure 2. Because the vector field defined by the system is parallel to the horizontal axis on the \(J\)-curve, and to the vertical axis on the right side of the square, where \(x_B = 1\), the solution curve cannot intersect \(\mathcal{M}\) these boundaries. Therefore, the solution will “leave” the \(\mathcal{M}\) region only through the segment BC on the upper horizontal side of the square. This intersection point is the required \(x_B\).

Step 2: Within-equilibrium deviations To show that within-range deviations are not profitable, consider \(\hat{p} > p\). We show that the slope of the expected profit function is negative at \(\hat{p}\). Denote \(x_B = X_B(p), x_S = X_S(p)\) and \(\hat{x}_B = X_B(\hat{p}), \hat{x}_S = X_S(\hat{p})\). Then the slope of the seller’s
expected profit $\Pi_S(x_S, \hat{p})$ is equal to
\[
\frac{\partial \Pi_S(x_S, \hat{p})}{\partial \hat{p}} = f_B(\hat{x}_B|x_S) \cdot \left( -x'_B(\hat{p})(\hat{p} - u_S(x_S, \hat{x}_B)) + \frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|x_S)} \right)
\]
Now from $u_S(x_S, x_B)$ being strictly increasing in $x_S$, and the MLRP, which implies
\[
\frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|x_S)} = \int_{\hat{x}_B}^{1} f_B(\hat{x}_B|x_S) \frac{d\hat{x}_B}{f_B(\hat{x}_B|x_S)} \leq \frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|x_S)},
\]
we see that
\[
\frac{\partial \Pi_S(x_S, \hat{p})}{\partial \hat{p}} \leq f_B(\hat{x}_B|x_S) \cdot \left( -x'_B(\hat{p})(\hat{p} - u_S(x_S, \hat{x}_B)) + \frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|x_S)} \right)
\]
But from our differential equation (7), the r.h.s. of the above inequality is 0. This shows that the expected profit has a non-positive slope for $\hat{p} > p$, and therefore such a deviation is not profitable.

**Step 3: Out-of-equilibrium deviations.** We now show that outside-range deviations are not profitable. We only need to consider price offers $p \in I \equiv [u_B(0,0), u_B(1,1)]$. This is because $p < u_B(0,0)$ would be acceptable to any buyer type regardless of the beliefs, and is therefore dominated for the seller by the price $p = u_B(0,0)$, while $p > u_B(1,1)$ will never be acceptable to any buyer type.

Our equilibrium does not restrict the buyer’s believe type of the seller as $\hat{x}_S$. In general, the buyer’s belief may be stochastic. Denote the distribution of $\hat{x}_S$ given price $p$ as $G_B(\hat{x}_S|p)$. Point beliefs are not ruled out, in which case the distribution $G_B$ is degenerate at a point. Given that we only consider a “serious” deviation, not higher than $u_B(1,1)$, there is always an interval of buyer types accepting such an offer. (This interval could be a single point if the offer is made exactly at $u_B(1,1)$.) Given a belief $\hat{x}_S$, we can alternatively reparameterize a deviation to a price $p > \bar{p}$ by the minimal buyer type $x$ for whom such a price is acceptable. The price itself is then $u_B(x, \hat{x}_S)$. Let $\Pi_S(x, x_S, \hat{x}_S)$ denote the seller’s profit if the buyer’s beliefs were known to the seller
\[
\Pi^*_S(x, x_S, \hat{x}_S) \equiv \int_{x}^{1} (u_B(x, \hat{x}_S) - u_S(\hat{x}_B, x_S)) f_B(\hat{x}_B|x_S) d\hat{x}_B,
\]
so that the expected seller’s profit from a deviation to $p$ is equal to $\int \Pi^*_S(x, x_S, \hat{x}_S) dG_B(\hat{x}_S|p)$. Since $\Pi^*_S(x, x_S, \hat{x}_S)$ is increasing in $\hat{x}_S$, the seller’s expected profit is bounded from above by $\Pi^*_S(x, x_S, 1)$. 

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For any \( x_S \in [0, 1] \), we have already ruled out within-range deviations, so
\[
\Pi(x_S, S(x_S)) \geq \Pi_S(x_S, p) = \Pi_S^*(x_B, x_S, 1),
\]
where the last equality follows by the fact that the offer \( p \) in equilibrium corresponds to \( x_S = 1 \): \( S(1) = p \).

The claim that there is no profitable deviation to \( p > p \) will follow once we show that the slope of the “most optimistic” profit function \( \Pi_S^*(x, x_S, 1) \) with respect to \( x \) is nonpositive. Now
\[
\frac{\partial \Pi_S(x, x_S, 1)}{\partial x} = - \left( u_B(x, 1) - u_S(x_S, x) - \frac{\partial u_B(x, 1)}{\partial x_B} \frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \right) \cdot f_B(x|x_S).
\]
Since the MLRP implies
\[
\frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \leq \frac{1 - F_B(x|1)}{f_B(x|1)},
\]
and \( u_S(x_S, x) < u_S(1, x) \), we have
\[
\begin{align*}
      & u_B(x, 1) - u_S(x_S, x) - \frac{\partial u_B(x, 1)}{\partial x_B} \frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \\
    & \geq u_B(x, 1) - u_S(1, x) - \frac{\partial u_B(x, 1)}{\partial x_B} \frac{1 - F_B(x|1)}{f_B(x|1)} \\
    & = J(x, 1) \geq 0,
\end{align*}
\]
where the last inequality follows because \( x > x_B \). This implies
\[
\frac{\partial \Pi_S^*(x, x_S, 1)}{\partial x} \leq 0,
\]
so even for the optimistic beliefs \( \hat{x}_S = 1 \), a deviation to \( p > \bar{p} \) is not profitable to the seller.

For a deviation to a price \( p < p \), we use the fact the assumed equilibrium belief upon such a deviation is \( \bar{x}_B = 0 \). We only need to consider the case \( \bar{x}_B > 0 \): if \( \bar{x}_B = 0 \), then \( p = u_B(0, 0) \) and the offer will be rejected by the buyer and is thus dominated by the (assumed) equilibrium offer. Also, we have already ruled out within-range deviations, so
\[
\Pi(x_S, S(x_S)) \geq \Pi(x_S, p) = \Pi_S^*(x_B, x_S, 0).
\]
A deviation to \( p \leq p \) is equivalent to picking \( x_B = x \in [0, \bar{x}_B] \), with the ensuing expected profit \( \Pi_S^*(x, x_S, 0) \). The slope of this profit function is
\[
\frac{\partial \Pi_S^*(x, x_S, 0)}{\partial x} = - \left( u_B(x, 0) - u_S(x_S, x) - \frac{\partial u_B(x, 0)}{\partial x_B} \frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \right) \cdot f_B(x|x_S).
\]
In this case, the MLRP implies
\[
\frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \geq \frac{1 - F_B(x|0)}{f_B(x|0)},
\]
while the monotonicity in own signal implies \( u_S(x_S, x) \geq u_S(0, x) \), and therefore the slope
\[
\frac{\partial \Pi^*_S(x, x_S, 0)}{\partial x} \geq -J(x, 0) \cdot f_B(x|x_S).
\]
Since \( J(x, 0) \leq J(x_B, 0) = 0 \), we see that
\[
\frac{\partial \Pi^*_S(x, x_S, 0)}{\partial x} \geq 0,
\]
so the profit is upward sloping for \( x \in [0, x_B] \) and a deviation to \( p < p \) is not profitable.

\[\square\]

3 Discussion

Some interesting implications of our results are discussed below.

**Signalling premium.** Recall that for a given \( x_S \), the marginal buyer’s type on the J-curve is denoted as \( x^*_B(x_S) \), and the price offer by the type \( x_S \) seller under revelation is \( S^0(x_S) \equiv u_B(x^*_B(x_S), x_S) \). Inspection of Figure 2 reveals that the solution (or equilibrium) curve, denoted as \( E \) in the graph, lies to the right of the J-curve that corresponds to the equilibrium under revelation. Since the marginal buyer’s type is related to the seller’s type as \( X_B(S(x_S)) \), this implies
\[
x^*_B(x_S) < X_B(S(x_S)) \quad \forall x_S \in (0, 1].
\]
By the strict monotonicity of \( u_B(x_{B}, x_{S}) \) in \( x_B \), we have
\[
u_B(x^*_B(x_S), x_S) < u_B(X_B(S^0(x_S)), x_S) \implies S^0(x_S) < S(x_S).
\]
In other words, the equilibrium seller’s offer under two-sided private information involves a **signalling premium** \( S(x_S) - S^0(x_S) > 0 \) for \( x_S \in (0, 1] \).
No regret. Any S-equilibrium satisfies the no-regret property for both buyers and sellers. That is, no buyer and no seller will ever obtain a negative ex-post surplus. For any price \( p \in [p, \bar{p}] \), i.e. any price at which trade can occur, \((X_B(p), X_S(p)) \in \mathcal{M}\), and therefore, as we have seen (refer to Figure 2), \( u_B(X_B(p), X_S(p)) = p \geq u_S(X_B(p), X_S(p)) \). For any seller type \( X_S(p) \) who in S-equilibrium trades with buyer types \( x \geq X_B(p) \), the surplus is at least \( p - u_S(x, X_S(p)) \) and is therefore non-negative. Similarly, for any buyer type \( X_B(p) \), who in S-equilibrium will trade with sellers having types \( x \leq X_S(p) \) and below, the surplus \( u_B(X_B(p), x) - p \) is non-negative.

Extension In this paper, we restrict attention to the case of one buyer. This is relevant in many applications. But there are also auction applications where the seller signals through posting a reserve price. Both Jullien and Mariotti (2006) and Cai, Riley, and Ye (2007) consider auctions. The presence of additional buyers poses further challenges in an affiliated environment as in this paper. In particular, the marginal profit no longer takes the simple form à la Myerson. This would be an interesting extension.

References


