

# Flippers in Housing Market Search – The Dynamics

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## Abstract

In this technical note, we study the dynamics of the “Flippers in Housing Market Search” model and analyze the local (in)stability of the steady-state equilibria.

## 1 The Dynamic Model

**Accounting Identities** The accounting identities

$$n_M + n_U + n_R = 1, \quad (1)$$

$$n_M + n_U + n_F = H, \quad (2)$$

and

$$\theta = \frac{S}{B} = \frac{n_U + n_F}{n_R} \quad (3)$$

hold in and off the steady state.

**Housing market flows** In a given time unit,  $\delta n_M$  matched households become unmatched. If a fraction  $\alpha \in [0, 1]$  of these households choose to sell right away in the investment market,

$$\dot{n}_M = \mu n_R - \delta n_M, \quad (4)$$

$$\dot{n}_U = (1 - \alpha) \delta n_M - \eta n_U, \quad (5)$$

$$\dot{n}_R = \alpha \delta n_M + \eta n_U - \mu n_R, \quad (6)$$

$$\dot{n}_F = \alpha \delta n_M - \eta n_F. \quad (7)$$

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Because mismatched homeowners at any moment in time can just sell right away to flippers,  $n_U$  may experience downward jumps at given moments when these individuals sell to flippers en masse. Denote the jumps as  $x \leq n_U$ . Then at times of jumps,

$$n_U^+ = n_U - x, \quad (8)$$

$$n_R^+ = n_R + x, \quad (9)$$

$$n_F = n_F + x, \quad (10)$$

where the superscript “+” denotes the value of the given variable right after the given moment in time.

**Market tightness** The two accounting identities (1) and (2) are two equations in four unknowns. Thus, it suffices to determine the time paths of any two of the four variables. Specifically, in terms of  $n_M$  and  $n_F$ , the system is completely described by (7) and

$$\dot{n}_M = \eta(H - n_M) - \delta n_M, \quad (11)$$

which is obtained from substituting  $\mu = \theta\eta$  and (3) into (4), with

$$\theta = \frac{S}{B} = \frac{n_U + n_F}{n_R} = \frac{H - n_M}{1 - H + n_F}. \quad (12)$$

Since  $\eta$  is a function of  $\theta$ , (7) and (11) is a system of two differential equations in  $n_M$  and  $n_F$ . In general, one cannot just analyze the dynamics of  $\theta$  alone because different combinations of  $n_M$  and  $n_F$  may give rise to the same  $\theta$ . In particular, the system is not in a steady state even when  $\theta$  happens to be equal to some particular steady-state value, if  $n_M$  and  $n_F$  are not at their respective steady states.

**Asset values** The value of a vacant house to a flipper is

$$r_F V_F = \eta(p_{IS} - V_F) + \dot{V}_F, \quad (13)$$

where under competition in real estate investment,

$$V_F = p_{FB}. \quad (14)$$

The asset value for a household in rental housing is

$$r_H V_R = -q + \mu(V_M - (\beta p_{FS} + (1 - \beta)p_H) - V_R) + \dot{V}_R, \quad (15)$$

where

$$\beta = \frac{n_F}{n_F + n_U} = \frac{n_F}{H - n_M} \quad (16)$$

denotes the fraction of houses for sale in the search market held by flippers. In the steady state,  $\beta = \alpha$ . Off steady state, the two need not be equal. A matched household has flow payoff equal to

$$r_H V_M = v + \delta (V_U - V_M) + \dot{V}_M. \quad (17)$$

A household in a mismatched house may sell the house right away to a flipper or to wait out a buyer to arrive in the search market. Thus for a small time interval  $d$ ,

$$V_U(t) = \max \left\{ V_R(t) + p_{FB}(t), \frac{1}{1 + r_H d} [\eta d (V_R(t+d) + p_H(t+d)) + (1 - \eta d) V_U(t+d) + o(d)] \right\},$$

where  $\lim_{d \rightarrow 0} o(d)/d = 0$ . To simplify, first, suppose that the first term inside the max operator is not smaller than the second term, in which case

$$V_U = V_R + p_{FB}.$$

On the other hand, if the second term dominates,

$$V_U(t) = \frac{1}{1 + r_H d} [\eta d (V_R(t+d) + p_H(t+d)) + (1 - \eta d) V_U(t+d) + o(d)].$$

Rearranging and taking limit,

$$r_H V_U = \eta (V_R + p_H - V_U) + \dot{V}_U.$$

In all,

$$V_U = \max \left\{ V_R + p_{FB}, \frac{1}{r_H} \left( \eta (V_R + p_H - V_U) + \dot{V}_U \right) \right\}. \quad (18)$$

**Households' maximization** Define

$$\Delta = V_R + p_{IB} - \frac{1}{r_H} \left( \eta (V_R + p_H - V_U) + \dot{V}_U \right). \quad (19)$$

The fraction of newly mismatched households who sell in the investment market is given by

$$\alpha = \begin{cases} 0 & \Delta < 0 \\ [0, 1] & \Delta = 0 \\ 1 & \Delta > 0 \end{cases}. \quad (20)$$

At moments in time at which  $\Delta$  changes sign from negative to positive,  $x = n_U = H - n_M - n_F$ . In any time interval in which  $\Delta = 0$  throughout, any sequence of  $x \leq n_U$  is potentially equilibrium.

**Bargaining** The two bargaining equations

$$V_M - p_{FS} - V_R = p_{FS} - V_F. \quad (21)$$

$$V_M - p_H - V_R = V_R + p_H - V_U. \quad (22)$$

should hold in and off the steady state.

**Dynamic Equilibrium** A dynamic equilibrium is the time paths for  $n_M$ ,  $n_F$ ,  $\theta$ ,  $V_F$ ,  $V_R$ ,  $V_M$ ,  $V_U$ ,  $p_{FB}$ ,  $p_{FS}$ ,  $p_H$ ,  $\Delta$ ,  $\alpha$ , and  $\beta$  that satisfy (10)-(22) and a sequence  $\{x\}$  where  $x = H - n_M - n_F$  during which  $\Delta$  changes sign from negative to positive, and that  $x \leq H - n_M - n_F$  during an interval in which  $\Delta = 0$ , given initial values for  $n_M$  and  $n_F$ . The values for  $n_U$  and  $n_F$  can then be recovered from (1) and (2).

## 2 The dynamics around a $\tilde{\theta}_L$ steady-state equilibrium

**Local stability** In a  $\tilde{\theta}_L$  steady-state equilibrium,  $\Delta > 0$  and  $\alpha = 1$ . By continuity, the same should hold in the neighborhood of the equilibrium. If selling in the investment market dominates selling in the search market, all mismatched homeowners would choose to sell to flippers right away. Then,  $n_U = 0$ , and (1) and (2) become

$$\begin{aligned} n_M + n_R &= 1, \\ n_M + n_F &= H, \end{aligned} \quad (23)$$

respectively. This is a system of two equations in three variables. Once any one of the three is pinned down, the other two are also uniquely determined. To proceed, solve the two equations for

$$n_R = 1 - H + n_F, \quad (24)$$

so that with  $n_U = 0$ ,

$$\theta = \frac{S}{B} = \frac{n_F}{n_R} = \frac{n_F}{1 - H + n_F}. \quad (25)$$

Thus, there is a one-to-one relation between  $\theta$  and  $n_F$ . Then, once the time path for  $\theta$  is known, so is that for  $n_F$  and therefore those for  $n_M$  and  $n_R$  as well. In all, in the special case of  $n_U = 0$ , it suffices to analyze the time path for  $\theta$ , and once  $\theta$  has reached a steady state, so did the other variables of interest. Now, to analyze the time path for  $\theta$ , solve (25) for

$$n_F = \frac{\theta(1-H)}{1-\theta}. \quad (26)$$

Differentiating,

$$\dot{n}_I = \frac{1-H}{(1-\theta)^2} \dot{\theta}. \quad (27)$$

With  $\alpha = 1$ , (7) becomes

$$\dot{n}_F = \delta n_M - \eta n_F = \delta (H - n_F) - \eta n_F, \quad (28)$$

where the second equal sign is from (23). Substituting (26) and (27) into (28)

$$\frac{1-H}{(1-\theta)^2} \dot{\theta} = \delta \left( H - \frac{\theta(1-H)}{1-\theta} \right) - \eta \frac{\theta(1-H)}{1-\theta}.$$

Simplifying,

$$\dot{\theta} = \frac{1-\theta}{1-H} (\delta (H - \theta) - \mu(\theta) (1-H)), \quad (29)$$

a first-order differential equation in  $\theta$ . By (25),  $\theta \in [0, H]$  and then by (29),

$$\dot{\theta} = \begin{cases} \frac{\delta H}{1-H} & \theta = 0 \\ -\mu(\theta)(1-H) & \theta = H \end{cases}.$$

Further, differentiating (29) with respect to  $\theta$ , factoring out  $1-H$ , and simplifying yields,

$$(2\theta - H - 1) \delta + (\mu - (1-\theta) \mu') (1-H),$$

which is increasing in  $\theta$ . Thus, the RHS of (29) is strictly convex in  $\theta$ . Given that it is positive at  $\theta = 0$  and negative at  $\theta = H$ , over the interval  $[0, H]$ , to the left of  $\tilde{\theta}_L$ ,  $\dot{\theta} > 0$  and to the right  $\dot{\theta} < 0$ . The  $\tilde{\theta}_L$  steady-state equilibrium is locally stable.

**The dynamics of vacancy and transaction** By (27),  $\dot{n}_F$  has the same sign as  $\dot{\theta}$ ; in the approach to the steady state, vacancy increases if and only if  $\theta$  increases. Transactions are comprised of houses sold to flippers by end users and houses sold to end users by flippers,

$$\begin{aligned} T &= \delta n_M + \eta n_F \\ &= \delta (H - n_F) + \eta n_F \\ &= \delta H + (1-H) (\eta - \delta) \frac{\theta}{1-\theta}, \end{aligned}$$

where the second and third lines are from (23) and (26), respectively. Differentiating,

$$\begin{aligned} \frac{\partial T}{\partial \theta} &= (1-H) \left( \frac{\theta}{1-\theta} \eta' + (\eta - \delta) \frac{1}{(1-\theta)^2} \right) \\ &= \frac{1-H}{1-\theta} \left( \theta \eta' + \frac{\eta - \delta}{1-\theta} \right), \end{aligned}$$

which generally cannot be signed.

**The dynamics of asset values and housing price** All transactions in the search market will be for houses sold from flippers to households in rental housing at price  $p_{FS}$  if all mismatched homeowners sell in the investment market in the first instance. By (21),

$$p_{FS} = \frac{V_M - V_R + V_F}{2}. \quad (30)$$

Substituting the equation into (13) and (15), respectively,

$$r_F V_F = \frac{\eta}{2} (V_M - V_R - V_F) + \dot{V}_F, \quad (31)$$

$$r_H V_R = -q + \frac{\mu}{2} (V_M - V_R - V_F) + \dot{V}_R, \quad (32)$$

Further, set  $V_U = V_R + p_{FB}$  in (17) and then substitute (14) for  $p_{FB}$  gives

$$r_H V_M = v - \delta (V_M - V_R - V_F) + \dot{V}_M. \quad (33)$$

Define  $V_D = V_M - V_R$ . By (32) and (33),

$$r_H V_D = v - q - \left( \delta + \frac{\mu}{2} \right) (V_D - V_F) + \dot{V}_D. \quad (34)$$

And then (31) becomes

$$r_F V_F = \frac{\eta}{2} (V_D - V_F) + \dot{V}_F, \quad (35)$$

Eqs (34) and (35) is a system of two differential equations in  $V_D$  and  $V_F$ , given  $\theta$ . Rewrite the system as

$$\dot{V}_D = \left( r_H + \delta + \frac{\mu}{2} \right) V_D - \left( \delta + \frac{\mu}{2} \right) V_F - v + q, \quad (36)$$

$$\dot{V}_F = \left( r_F + \frac{\eta}{2} \right) V_F - \frac{\eta}{2} V_D. \quad (37)$$

Suppose  $\theta = \tilde{\theta}_L$ . The steady state is given by the pair of equations

$$\dot{V}_D = 0 : \quad V_F = \frac{\left( r_H + \delta + \frac{\mu(\tilde{\theta}_L)}{2} \right) V_D - v + q}{\left( \delta + \frac{\mu(\tilde{\theta}_L)}{2} \right)}, \quad (38)$$

$$\dot{V}_F = 0 : \quad V_F = \frac{\frac{\eta(\tilde{\theta}_L)}{2} V_D}{\left( r_F + \frac{\eta(\tilde{\theta}_L)}{2} \right)}. \quad (39)$$

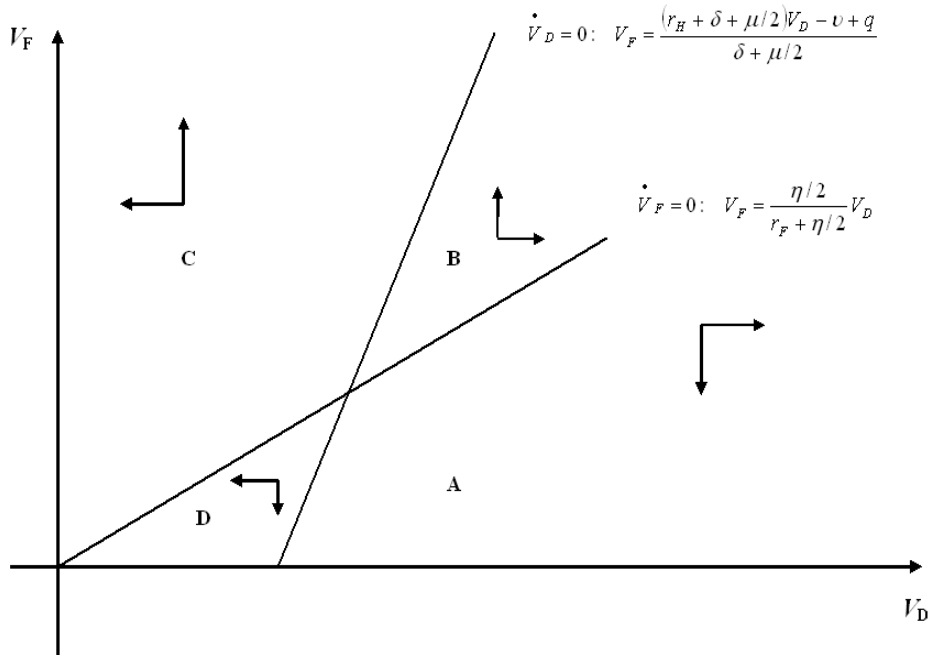


Figure 1: The dynamics of  $V_D$  and  $V_F$  with a stationary  $\theta$

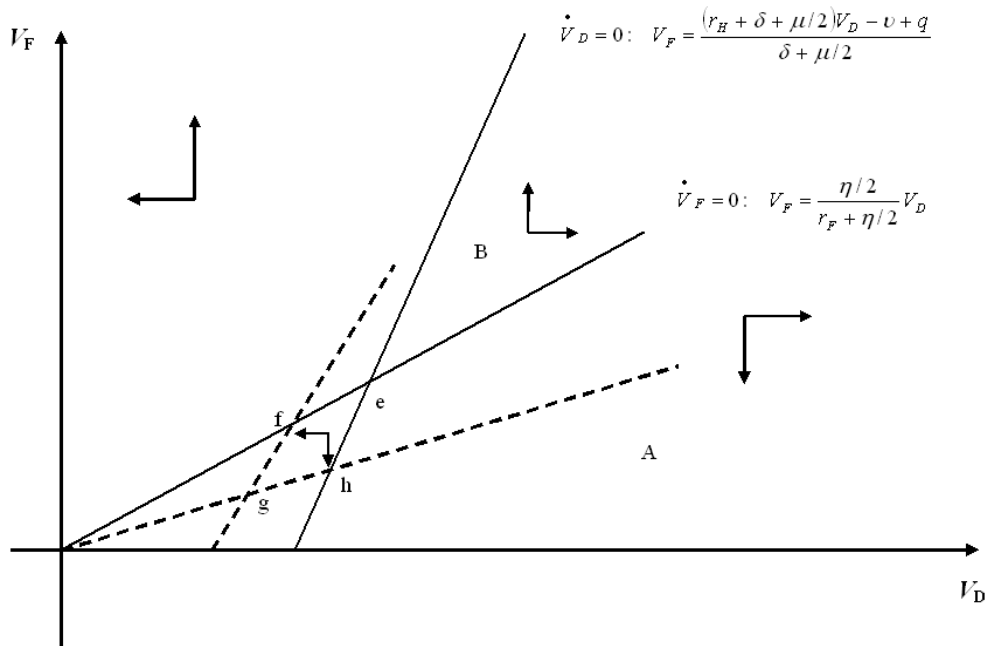


Figure 2: The dynamics of  $V_D$  and  $V_F$  with a moving  $\theta$

Figure 1 shows how the pair  $(V_D, V_F)$  will immediately jump to their respective steady states given that the system (36) and (37) is everywhere unstable.

Next, consider the dynamics of  $V_D$  and  $V_F$  to accompany a changing  $\theta$ . Without loss of generality, consider where initially  $\theta = \theta_0 < \tilde{\theta}_L$ , so that  $\dot{\theta} > 0$ . Where  $n_U = 0$ , by (25), if  $\dot{\theta} > 0$ , then  $\dot{n}_F > 0$  as well; by (23) and (24), respectively, if  $\dot{n}_F > 0$ ,  $\dot{n}_M < 0$  and  $\dot{n}_R > 0$ . As  $\theta$  increases towards  $\tilde{\theta}_L$ ,  $\eta(\theta)$  falls, which flattens the  $\dot{V}_F = 0$  line in Figure 1, while  $\mu(\theta)$  increases, which steepens the  $\dot{V}_D = 0$  line and lowers the horizontal intercept in the Figure. In Figure 2, the solid lines depict the  $\dot{V}_F = 0$  and  $\dot{V}_D = 0$  lines for  $\theta = \theta_0$ , whereas the dotted lines are for  $\theta = \tilde{\theta}_L$ . We seek to determine where the  $(V_D, V_F)$  pair lies when  $\theta = \theta_0$  and the dynamics of the pair as  $\theta$  increases toward  $\tilde{\theta}_L$ . Specifically, at the moment  $\theta = \theta_0$ , the  $(V_D, V_F)$  pair must lie in the interior of the area bounded by  $efgh$  for the pair to eventually reach the steady state at point  $g$ . The following arguments exclude all other possibilities.

1. At  $e$ ,  $\dot{V}_F = 0$  and  $\dot{V}_D = 0$ ; in the next instant as  $\theta$  increases, the  $(V_D, V_F)$  pair will be in region  $B$ .
2. At  $f$ ,  $\dot{V}_F = 0$  and  $\dot{V}_D < 0$ ; in the next instant, the  $(V_D, V_F)$  pair will be in region  $C$ .
3. At  $g$ ,  $\dot{V}_F < 0$  and  $\dot{V}_D < 0$ ; in the next instant, the  $(V_D, V_F)$  pair will be outside of  $efgh$ .
4. At  $h$ ,  $\dot{V}_F < 0$  and  $\dot{V}_D = 0$ ; in the next instant, the  $(V_D, V_F)$  pair will be in region  $A$ .
5. Between  $e$  and  $f$ ,  $\dot{V}_F = 0$  and  $\dot{V}_D < 0$ ; in the next instant as  $\theta$  increases, the  $(V_D, V_F)$  pair will be in either region  $B$  or  $C$ .
6. Between  $f$  and  $g$  and between  $g$  and  $h$ ;  $\dot{V}_F < 0$  and  $\dot{V}_D < 0$ ; in the next instant, the  $(V_D, V_F)$  pair will be outside of  $efgh$ .
7. Between  $e$  and  $h$ ,  $\dot{V}_D = 0$  and  $\dot{V}_F < 0$ ; in the next instant, the  $(V_D, V_F)$  pair will be in either region  $A$  or  $B$ .

Given that the  $(V_D, V_F)$  pair must lie strictly within the area bounded by  $efgh$ , in the approach to the steady state,  $\dot{V}_D < 0$  and  $\dot{V}_F < 0$ . To verify uniqueness (saddlepoint stability), we examine the three equation system: (29), (36), and (37),

the Jacobian of which is

$$J_E = \begin{bmatrix} \frac{(2\theta-H-1)}{1-H}\delta + (\mu - (1-\theta)\mu') & 0 & 0 \\ \frac{\mu'}{2}(V_D - V_F) & r_H + \delta + \frac{\mu}{2} & -\delta - \frac{\mu}{2} \\ \frac{\eta'}{2}(V_F - V_D) & -\frac{\eta}{2} & r_F + \frac{\eta}{2} \end{bmatrix},$$

yielding a determinant equal to

$$|J_E| = \left( \frac{(2\theta - H - 1)}{1 - H} \delta + (\mu - (1 - \theta) \mu') \right) \left[ \frac{\eta}{2} r_H + \frac{\mu}{2} r_F + \delta r_F + r_H r_F \right] < 0,$$

since around the steady state,

$$\frac{\partial \dot{\theta}}{\partial \theta} = \frac{(2\theta - H - 1)}{1 - H} \delta + (\mu - (1 - \theta) \mu') < 0.$$

With  $\theta$  a backward-looking stock variable and the three asset values forward-looking jump variables, there is a unique equilibrium path for the triple  $(V_M, V_F, V_R)$  for any given  $\theta_0$ .

Finally, with both  $\dot{V}_D < 0$  and  $\dot{V}_F < 0$  in the approach to the steady state from below, by (14) and (30), respectively,  $\dot{p}_{FS} < 0$  and  $\dot{p}_{FB} < 0$ . Intuitively, an increase in  $\theta = S/B$  weakens sellers' bargaining power and it is only reasonable that lower prices follow.

**Price, transaction, and vacancy** In all, in the transition to the  $\tilde{\theta}_L$  steady-state equilibrium with a  $\theta < \tilde{\theta}_L$ , so that there are increasingly more houses in the hands of flippers in the midst of more and more households staying in rental housing, housing prices fall while vacancy rises. Meanwhile, transaction volume can change in the same or in the opposite direction. To summarize,

**Proposition 1** *A  $\tilde{\theta}_L$  steady-state equilibrium is locally stable and the transition should be direct. Price and vacancy are negatively correlated in the approach to the steady state, whereas there exists no definite relation between the two variables on the one hand and transaction volume on the other hand.*

### 3 The dynamics around a $\tilde{\theta}_U$ steady-state equilibrium

**Local stability** In a  $\tilde{\theta}_U$  steady-state equilibrium,  $\Delta < 0$  and  $\alpha = 0$ . By continuity, the same should hold in the neighborhood of the equilibrium. Also, in a  $\tilde{\theta}_U$  steady-state equilibrium,  $n_F = 0$ . The same, however, need not hold in the neighborhood of the equilibrium. True, with  $\Delta < 0$ , selling in the search market dominates selling

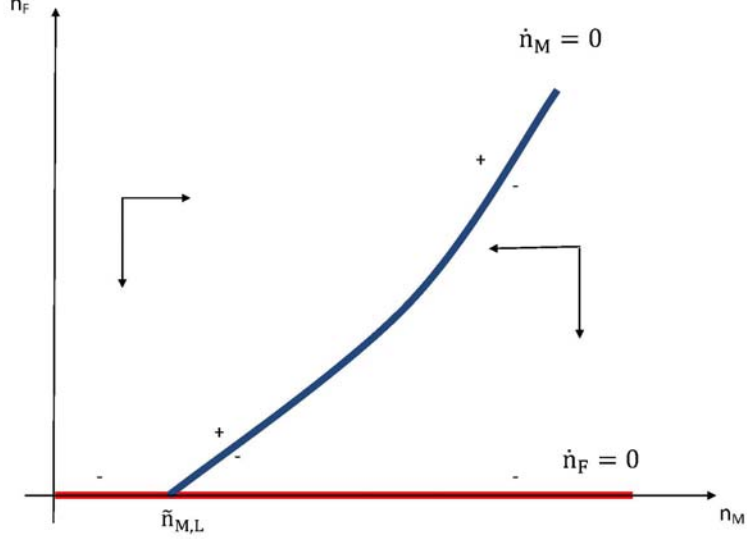


Figure 3: The dynamics of  $n_M$  and  $n_F$  with  $\alpha = 0$

in the investment market for mismatched homeowners, and no houses will be sold to flippers. But it takes time for flippers to dispose of their inventories (if any) in the search market so that  $n_F$  can remain positive in the approach to the  $\tilde{\theta}_U$  steady-state equilibrium. Around a  $\tilde{\theta}_L$  steady-state equilibrium, where  $n_U = 0$ , the dynamics can be summarized by one differential equation in  $\theta$ , as we demonstrated in the last section. Here with both  $n_M$  and  $n_F$  unrestricted, we must work with the system of differential equations in (7) and (11).

To begin, in the  $\tilde{\theta}_U$  steady state,

$$\dot{n}_I = 0 : n_F = 0,$$

$$\dot{n}_M = 0 : \eta(H - n_M) - \delta n_M = 0.$$

Totally differentiating the last equation,

$$\frac{\partial n_F}{\partial n_M} = -\frac{-\theta\eta' - \eta - \delta}{-\theta^2\eta'},$$

which has the same sign as

$$\theta\eta' + \eta + \delta = \frac{\theta\mu' - \mu}{\theta} + \frac{\mu}{\theta} + \delta > 0.$$

Furthermore,

$$\frac{\partial \dot{n}_M}{\partial n_M} = -\theta\eta' - \eta - \delta < 0.$$

The phase diagram is depicted in Figure 3, where the steady state is  $(n_F, n_M) = (0, \tilde{n}_{M,L})$ , which corresponds to  $\theta = \tilde{\theta}_U$ . In the convergence to the steady state,  $\dot{n}_F < 0$  but  $\dot{n}_M$  can be positive or negative. In particular,

1. For  $n_M < \tilde{n}_{M,L}$ ,  $\dot{n}_M > 0$  must hold.
2. For  $n_M \geq \tilde{n}_{M,L}$ ,  $\dot{n}_M \leq 0$  only for sufficiently small  $n_F$  (relative to  $n_M$ ), i.e., the  $(n_F, n_M)$  pair is on the RHS of the  $\dot{n}_M = 0$  locus. For larger  $n_F$ , so that the  $(n_F, n_M)$  pair is on the LHS of the  $\dot{n}_M = 0$  locus,  $\dot{n}_M > 0$  and will remain so until the pair crosses the locus. Once the pair is on the RHS of the locus, it must remain there, until the steady state is attained.

That is, the  $\tilde{\theta}_U$  steady state is everywhere stable, though the convergence may not always be direct. When the convergence is not direct, which happens when the  $(n_F, n_M)$  pair is sufficiently far from the  $\tilde{\theta}_U$  steady state, the convergence of  $\theta$  to  $\tilde{\theta}_U$  may oscillate. Eventually when  $n_F$  is nearing zero, the approach must become direct. In Figure 3, with  $n_F = 0$ ,  $\dot{n}_M \geq 0$  if and only if  $n_M \leq \tilde{n}_{M,L}$ . And then with

$$\theta = \frac{H - n_M}{1 - H}, \quad (40)$$

where  $n_F = 0$ ,  $\theta \geq \tilde{\theta}_U$  and  $\dot{\theta} \leq 0$  if and only if  $n_M \leq \tilde{n}_{M,L}$  and  $\dot{n}_M \geq 0$ , respectively. In all, for  $n_M \leq \tilde{n}_{M,L}$ ,  $\theta \geq \tilde{\theta}_U$ ; meanwhile,  $\dot{n}_M \geq 0$  and  $\dot{\theta} \leq 0$ . For simplicity, in the analysis of the dynamics of vacancy, transaction, and housing price in the following, we shall assume the direct convergence of  $\theta$  to  $\tilde{\theta}_U$  due to  $n_F \simeq 0$ .

**The dynamics in vacancy and transaction** With  $n_F \simeq 0$ , housing market transactions are dominated by sales and purchases between a mismatched homeowner and a household in rental housing. Transaction volume is

$$T \simeq n_R \mu \simeq (1 - H) \mu,$$

which is an increasing function of  $\theta$ . Vacancy is negligible given  $n_F \simeq 0$ , so that (almost) all houses are owned and occupied by end users.

**The dynamics in asset values and housing price** If the investment market is inactive, there will be just one price at which houses are bought and sold—the price in the search market. Solve (22) for

$$p_H = \frac{V_M + V_U}{2} - V_R. \quad (41)$$

Substitute it into (15) and (18), respectively, while setting  $\beta = 0$  in the former and noting that  $\Delta < 0$  in the latter,

$$r_H V_R = -q + \mu \frac{V_M - V_U}{2} + \dot{V}_R, \quad (42)$$

$$r_H V_U = \eta \frac{V_M - V_U}{2} + \dot{V}_U. \quad (43)$$

Eqs (17) and (42)-(43) is a system of three differential equations in  $V_M$ ,  $V_R$ , and  $V_U$ .

To proceed, note that the system can be divided into two blocks: (17) and (43) in  $V_M$  and  $V_U$  alone and then once  $V_M$  and  $V_U$  are determined,  $V_R$  follows from (42). For the first block, rewrite the equations as

$$\dot{V}_U = \left(r_H + \frac{\eta}{2}\right) V_U - \frac{\eta}{2} V_M, \quad (44)$$

$$\dot{V}_M = (r_H + \delta) V_M - \delta V_U - v. \quad (45)$$

Now suppose that  $\theta$  has reached the  $\tilde{\theta}_U$  steady state, so that  $\eta$  is stationary over time. The steady states of  $V_U$  and  $V_M$  solve the system

$$\dot{V}_U = 0 : V_U = \frac{\frac{\eta(\tilde{\theta}_U)}{2} V_M}{r_H + \frac{\eta(\tilde{\theta}_U)}{2}}, \quad (46)$$

$$\dot{V}_M = 0 : V_U = \frac{(r_H + \delta) V_M - v}{\delta}. \quad (47)$$

The phase diagram is depicted in Figure 4. The system  $(V_M, V_U)$  is everywhere unstable. In a rational expectations equilibrium, given a stationary  $\theta$ ,  $\dot{V}_M = \dot{V}_U = 0$  must attain immediately.

Next, consider the dynamics of  $V_M$  and  $V_U$  to accompany a changing  $\theta$ . Consider where initially  $\theta_0 < \tilde{\theta}_U$  and assume that  $\dot{\theta} > 0$  holds. By (2) with  $n_F \simeq 0$  and (40),  $\dot{n}_M < 0$  and  $\dot{n}_U > 0$ . As  $\theta$  increases over time towards  $\tilde{\theta}_U$ ,  $\eta(\theta)$  falls, which will flatten the  $\dot{V}_U = 0$  line in Figure 4. The  $\dot{V}_M = 0$  line remains unaffected, however, since  $\dot{V}_M$  is independent of  $\theta$ . Figure 5 plots two  $\dot{V}_U = 0$  lines, for  $\theta = \theta_0$  and  $\theta = \tilde{\theta}_U$ , respectively, together with the given  $\dot{V}_M = 0$ . We seek to determine where the  $(V_U, V_M)$  pair lies when  $\theta = \theta_0$  and the dynamics of the pair as  $\theta$  increases toward  $\tilde{\theta}_U$ . Now, clearly the  $(V_U, V_M)$  pair cannot lie within regions  $A$ ,  $B$ , nor  $C$ ; for otherwise it cannot eventually settle at the steady state, depicted as point  $g$  in the Figure. The only remaining possibilities are then points within region  $D$ , which are also to the northeast of  $g$ —the area bounded by  $efg$ , because only then can the pair be moving towards  $g$  over time. Furthermore, we can rule out points on the boundaries of the triangle  $efg$ :

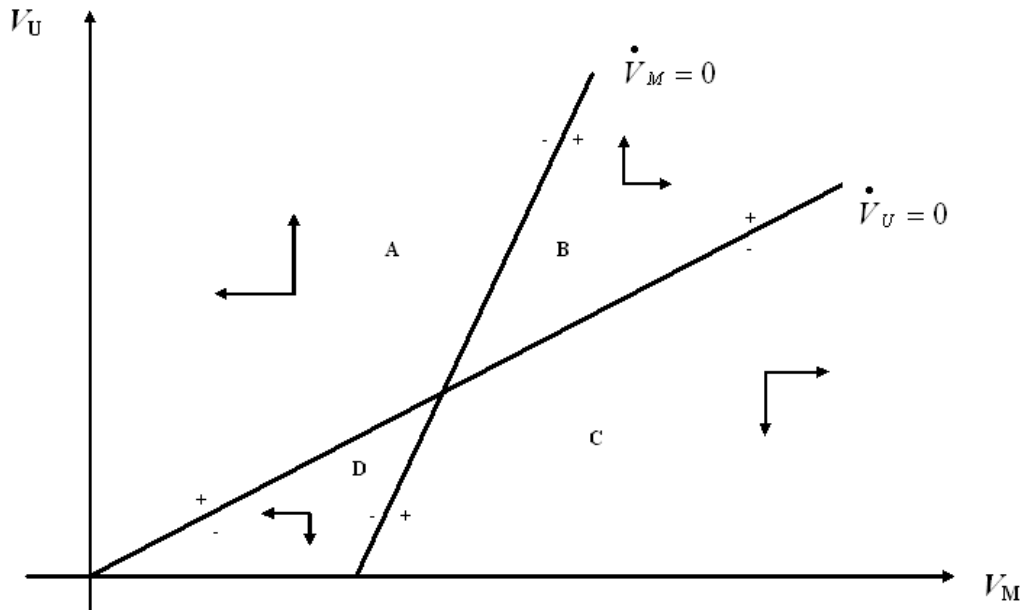


Figure 4: The Dynamics of  $V_U$  and  $V_M$  with a stationary  $\theta$

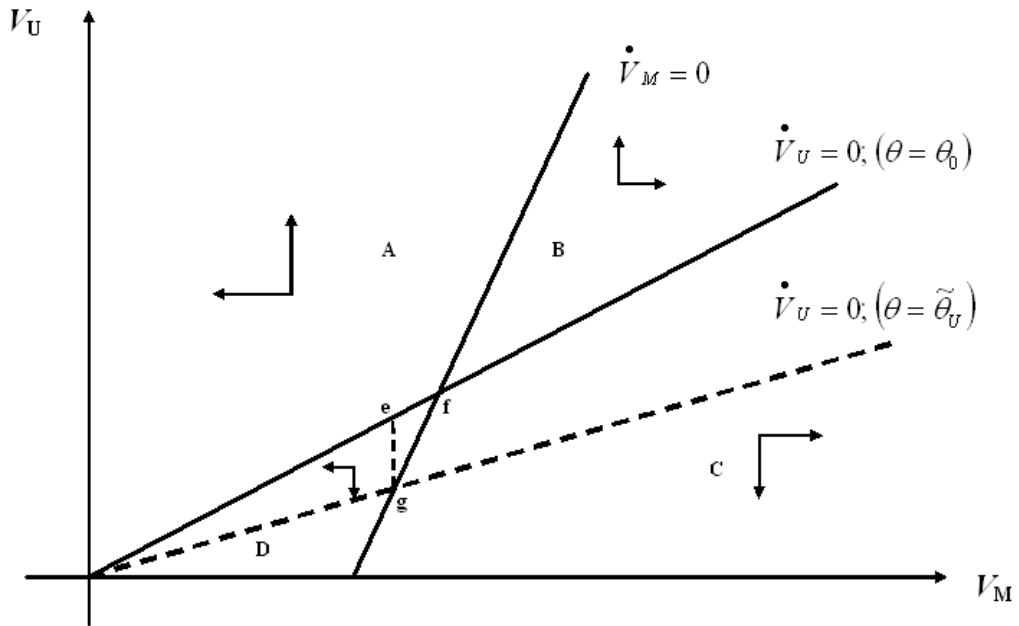


Figure 5: The Dynamics of  $V_M$  and  $V_U$  with a moving  $\theta$

1. At  $g$ ,  $\dot{V}_M = 0$  and  $\dot{V}_U < 0$ ; in the next instant, the  $(V_U, V_M)$  pair will be in region  $C$ .
2. At  $e$ ,  $\dot{V}_U = 0$  and  $\dot{V}_M < 0$ ; in the next instant, the  $(V_U, V_M)$  pair will be in region  $A$ .
3. At  $f$ ,  $\dot{V}_U = 0$  and  $\dot{V}_M = 0$ ; in the next instant as  $\theta$  increases,  $\dot{V}_U > 0$ , the  $(V_U, V_M)$  pair will be in region  $A$ .
4. Between  $e$  and  $f$ ,  $\dot{V}_U = 0$  and  $\dot{V}_M < 0$ ; in the next instant as  $\theta$  increases,  $\dot{V}_U > 0$ , the  $(V_U, V_M)$  pair will be in region  $A$ .
5. Between  $e$  and  $g$ ,  $\dot{V}_U < 0$  and  $\dot{V}_M < 0$ ; in the next instant, the  $(V_U, V_M)$  pair will be outside of  $efg$ .
6. Between  $f$  and  $g$ ,  $\dot{V}_M = 0$  and  $\dot{V}_U < 0$ ; in the next instant, as  $\theta$  increases and the  $\dot{V}_U = 0$  line shifts down, the  $(V_U, V_M)$  pair will be either in region  $B$  or  $C$ .

The  $(V_U, V_M)$  pair must then lie strictly within the  $efg$  triangle, and therefore in the approach to the steady state,  $\dot{V}_M < 0$  and  $\dot{V}_U < 0$ . To verify uniqueness (saddlepoint stability), first rewrite (40) as

$$n_M = H - \theta(1 - H), \quad (48)$$

and differentiating,

$$\dot{n}_M = -\dot{\theta}(1 - H). \quad (49)$$

Combining (11), (48), and (49) yields

$$\dot{\theta} = \delta \left( \frac{H}{1 - H} - \theta \right) - \mu, \quad (50)$$

the differential equation in  $\theta$  in case  $n_F = 0$ . The system (50), (42), (44), and (45) is a system of four differential equations in  $\theta$ ,  $V_R$ ,  $V_U$ , and  $V_M$ , the Jacobian of which is

$$J_E = \begin{bmatrix} -\delta - \mu' & 0 & 0 & 0 \\ \frac{\eta'}{2}(V_U - V_M) & r_H + \frac{\eta}{2} & -\frac{\eta}{2} & 0 \\ 0 & -\delta & r_H + \delta & 0 \\ \frac{\mu'}{2}(V_U - V_M) & \frac{\mu}{2} & -\frac{\mu}{2} & r \end{bmatrix},$$

yielding a determinant,

$$|J_E| = -(\delta + \mu') r_H^2 \left( r_H + \delta + \frac{1}{2}\eta \right) < 0.$$

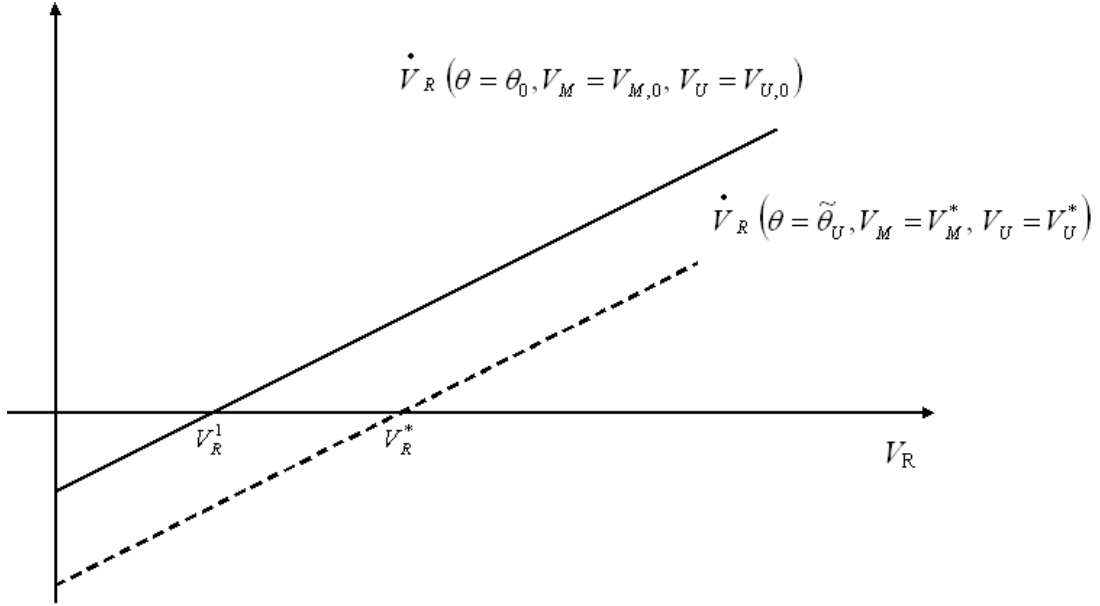


Figure 6: The dynamics of  $V_R$

With a negative determinant, the system exhibits saddlepoint stability. With  $\theta$  a backward-looking stock variable and the three asset values forward-looking jump variables, there is a unique equilibrium path for the triple  $(V_M, V_U, V_R)$  for any given  $\theta_0$ .

Now, we know that if  $\dot{\theta} > 0$ ,  $\dot{V}_M < 0$  and  $\dot{V}_U < 0$ . In this case, an examination of (45) reveals that in the approach to the steady state,  $V_M - V_U$  must increase over time. And then by (42), the same tendency must hold for  $V_R$ . To see this, suppose initially  $V_M = V_{M,0}$  and  $V_U = V_{U,0}$  when  $\theta = \theta_0$ . In Figure 6, if  $V_{R,0}$  is to the left of  $V_R^1$ ,  $\dot{V}_R < 0$ , given the initial values for  $\theta$ ,  $V_M$ , and  $V_U$ . Over time as both  $\theta$  and  $V_M - V_U$  increase, by (42), the  $\dot{V}_R$  line shifts down and to the right;  $\dot{V}_R < 0$  must continue to hold. Hence for  $V_R$  to reach the steady state  $V_R^*$ ,  $V_{R,0}$  must lie to the right of  $V_R^1$ , and that  $\dot{V}_R > 0$ . Thus, with  $\dot{V}_R > 0$  but  $\dot{V}_M < 0$  and  $\dot{V}_U < 0$ , by (41), housing price falls in the approach to the steady state given that initially  $\theta_0 < \tilde{\theta}_U$ .

**Price, transaction, and vacancy** In all, in the transition to the  $\tilde{\theta}_U$  steady-state equilibrium with a  $\theta < \tilde{\theta}_U$  and if flippers' inventories are negligible, while there are increasingly more mismatched and fewer matched households, housing prices fall and transaction volume increases. To summarize,

**Proposition 2** *The  $\tilde{\theta}_U$  steady-state equilibrium is locally stable but the transition can oscillate until the  $(n_F, n_M)$  pair is nearing the given steady state. In the transition*

to the given equilibrium and if  $n_F \simeq 0$ , price and transaction volume are negatively correlated, whereas vacancy stays constant throughout.

## 4 The dynamics around $\widehat{\theta}$ , $\widehat{\theta}_1$ , and $\widehat{\theta}_2$ steady-state equilibrium

In a  $\widehat{\theta}$ ,  $\widehat{\theta}_1$ , or  $\widehat{\theta}_2$  steady-state equilibrium,  $\Delta = 0$ . It does not seem possible to rule out a priori by any continuity argument whether or not the same equality of payoff holds around the given steady state. To proceed, we first assume  $\Delta = 0$  also holds in the approach to the given steady state and then repeat the analysis for otherwise.

### 4.1 $\Delta = 0$ in the approach to the steady state

If the payoff to selling in the search market is not lower than that in the investment market, the equation for  $V_U$  in (43) remains valid. If the payoffs to selling in the two markets are indeed equal; i.e.,

$$V_U = V_R + p_{FB} = V_R + V_F, \quad (51)$$

the equation for  $V_R$  in (42) holds as well. Besides, the equation for  $V_F$  in (31) becomes

$$r_F V_F = \frac{\eta}{2} (V_M - V_U) + \dot{V}_I. \quad (52)$$

Now, add (52) to (42), rearrange, and by (51),

$$r_H V_U = - (r_F - r_H) V_F - q + \frac{\mu + \eta}{2} (V_M - V_U) + \dot{V}_U.$$

Equate this equation with (43) and simplify,

$$0 = \frac{\mu}{2} (V_M - V_U) - (r_F - r_H) V_F - q.$$

Let  $V_D = V_M - V_U$ ; the above becomes

$$V_D = 2 \frac{(r_F - r_H) V_F + q}{\mu}. \quad (53)$$

Differentiating,

$$\dot{V}_D = 2 \frac{(r_F - r_H) \dot{V}_I \mu - \mu' \dot{\theta} ((r_F - r_H) V_F + q)}{\mu^2}. \quad (54)$$

Next, subtract (43) from (17) and with  $V_D = V_M - V_U$ ,

$$\dot{V}_D = \left( r_H + \delta + \frac{\eta}{2} \right) V_D - v. \quad (55)$$

Set the RHSs of (54) and (55) equal and solve the equation for

$$\dot{\theta} = \frac{(r_F - r_H)\mu}{\mu'((r_F - r_H)V_F + q)} \dot{V}_I - \frac{\mu}{\mu'} \left( r_H + \delta + \frac{\eta}{2} \right) + \frac{\mu^2}{2\mu'((r_F - r_H)V_F + q)} v. \quad (56)$$

Combine (52) and (53) and rearrange,

$$\dot{V}_F = \left( r_F - \frac{(r_F - r_H)}{\theta} \right) V_F - \frac{q}{\theta}. \quad (57)$$

Substitute (57) into (56) and simplify,

$$\begin{aligned} \dot{\theta} = \frac{\mu}{\mu'} \left\{ \frac{r_F - r_H}{(r_F - r_H)V_F + q} \left( \left( r_F - \frac{r_F - r_H}{\theta} \right) V_F - \frac{q}{\theta} \right) - \left( r_H + \delta + \frac{\mu}{2\theta} \right) \right. \\ \left. + \frac{\mu}{2} \frac{v}{(r_F - r_H)V_F + q} \right\}. \end{aligned} \quad (58)$$

Eqs (57) and (58) is a system of two first-order differential equations in  $V_F$  and  $\theta$ , which describes how the two variables must evolve to maintain the equality of payoffs between selling in the two markets for households. Around any steady state to which the  $(\theta, V_F)$  pair approaches,  $\dot{V}_F \simeq 0$  and by (57),

$$V_F \simeq \frac{q}{\theta r_F - (r_F - r_H)},$$

the positivity of which requires

$$r_F - \frac{r_F - r_H}{\theta} > 0.$$

Hence around the given steady state,

$$\frac{\partial \dot{V}_F}{\partial V_F} = r_F - \frac{r_F - r_H}{\theta} > 0.$$

Where  $\theta$  is a backward-looking stock variable and  $V_F$ , as an asset value, a forward-looking jump variable and that  $\partial \dot{V}_F / \partial V_F > 0$ , in a rational expectations equilibrium,  $V_F$  will immediately jump to its steady-state value where  $\theta$  is in the steady state. By extension, there will be a unique path of  $(\theta, V_F)$  in the approach to a given steady state if there should be convergence at all.

The actual evolution of  $\theta$ , however, is governed by another system of differential equations in (7) and (11). Presumably, the first system in (57) and (58) is equilibrium if there exists a time path for  $\alpha \in [0, 1]$  that makes the second system imply the same unique time path for  $\theta$  from the first system. An inspection of (7) and (11), together with (12), reveals that there is at most one  $\alpha$  time path that will make  $\theta$  travel along

a given path from an initial  $\theta_0$  implied by the initial  $(n_M, n_F)$  pair. When  $\theta$  hits any given steady state, however, there is no guarantee that the  $(n_M, n_F)$  pair is at its steady state. In general, it will not. Within the system (7) and (11), the movement of one free variable  $\alpha$  can only take one variable,  $\theta$ , to follow a given time path, but not two variables,  $\theta$  and either  $n_F$  or  $n_M$ , to follow two respective time paths. A priori, since  $\Delta = 0$  is assumed to hold throughout, there can also be upward jumps in  $n_F$  up to  $H - n_M - n_F$  to aid the evolution of  $(n_M, n_F)$  to its steady state. However, any jumps in  $n_F$  are also jumps in  $\theta$ , which contradicts the continuity of  $\theta$  implied by (57) and (58). In all, any dynamic equilibrium where  $\Delta = 0$  holds off the steady state is almost always divergent. Any convergence must be sheer coincidence.

## 4.2 $\Delta \neq 0$ in the approach to the steady state

### 4.2.1 $\Delta < 0$ in the approach to the steady state

If  $\Delta < 0$  holds in the approach to a steady state, in the interim,  $\alpha = 0$ . The dynamics of the  $(n_F, n_M)$  pair and of the implied  $\theta$  are as described in Section 3 and Figure 3. Eventually, the convergence is to a  $\tilde{\theta}_U$  steady-state equilibrium if  $\Delta < 0$  indeed holds throughout. But in certain situations,  $\Delta$  must change sign sooner or later. To proceed, first consider the dynamics of the  $(n_F, n_M)$  pair for some  $\alpha \in (0, 1]$ . The  $\dot{n}_M = 0$  locus remains given by that in Figure 3, whereas by (7), the  $\dot{n}_F = 0$  locus generalizes to

$$\alpha\delta n_M - \eta n_F = 0. \quad (59)$$

On this locus, at  $n_M = 0$ ,  $n_F = 0$ ; as  $n_M \rightarrow H$ ,  $n_F \rightarrow 0$  as well since with  $n_M = H$ ,  $\theta = 0$  and  $\eta \rightarrow \infty$ . Furthermore, totally differentiating gives

$$\frac{\partial n_F}{\partial n_M} = -\frac{\alpha\delta + \frac{n_F}{1-H+n_F}\eta'}{\frac{H-n_M}{(1-H+n_F)^2}\eta' - \eta}.$$

The denominator is negative and thus the above has the same sign as

$$\alpha\delta + \frac{n_F}{1-H+n_F}\eta'.$$

Differentiate with respect to  $n_M$  one more time,

$$-\frac{n_F}{(1-H+n_F)^2}\eta'' < 0,$$

given that  $\eta'' > 0$ . In sum, the  $\dot{n}_F = 0$  locus in (59) must be strictly concave with  $n_F = 0$  where  $n_M = 0$  and  $H$ . See Figure 7. One implication is that so long as  $\alpha$  remains constant, the steady state is stable, though the convergence can be oscillatory.

Next consider an increase in  $\alpha$ . By (59),

$$\frac{\partial n_F}{\partial \alpha} = -\frac{\delta n_M}{\frac{H-n_M}{(1-H+n_F)^2}\eta' - \eta} > 0.$$

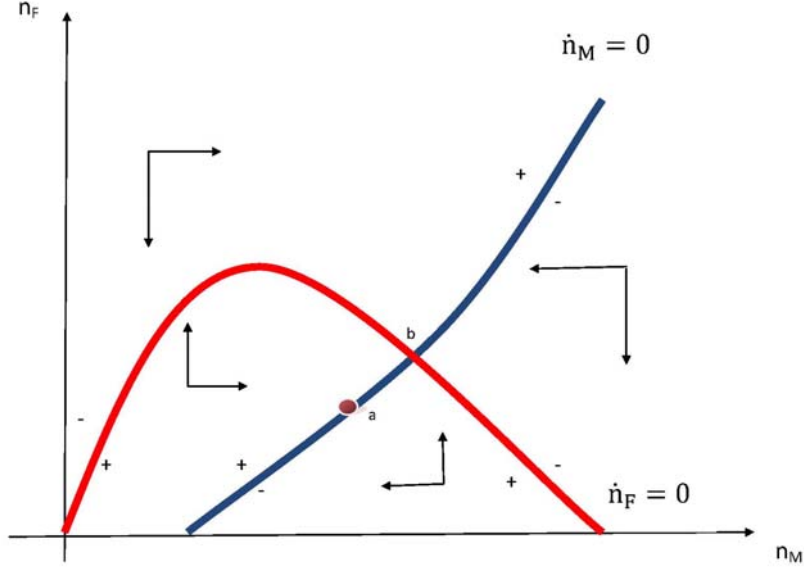


Figure 7: The dynamics of  $n_M$  and  $n_F$  with  $\alpha > 0$

Thus, as  $\alpha$  increases, the  $\dot{n}_F = 0$  locus will intersect the  $\dot{n}_M = 0$  locus at larger and larger  $(n_F, n_M)$  pairs.

In Figure 8, point  $a$  is a steady-state equilibrium with some  $\alpha \in (0, 1)$ . Denote the steady-state pair as  $(\tilde{n}_F(\alpha), \tilde{n}_M(\alpha))$ . Suppose off this steady state,  $\alpha = 0$ , then the  $(n_F, n_M)$  pair will return to  $(\tilde{n}_F(\alpha), \tilde{n}_M(\alpha))$  only if (i)  $n_F > \tilde{n}_F(\alpha)$  and (ii) and  $n_M$  is not too far off from the  $\dot{n}_M = 0$  locus relative to the deviation of  $n_F$  from  $\tilde{n}_F(\alpha)$ . For any other  $(n_F, n_M)$  pair, the convergence is to the  $(n_F, n_M) = (0, \tilde{n}_{M,L})$  steady state with  $\theta = \tilde{\theta}_U$ . The first condition is intuitive. Since  $\dot{n}_F < 0$  with  $\alpha = 0$ , if  $n_F < \tilde{n}_F(\alpha)$  to begin with, it cannot approach  $\tilde{n}_F(\alpha)$ .

In sum, around an  $\alpha \in (0, 1)$  steady-state equilibrium and if  $\alpha = 0$  holds, there will be convergence if  $n_F > \tilde{n}_F(\alpha)$  and that the deviation of  $n_M$  from  $\tilde{n}_M(\alpha)$  is small. Otherwise, the convergence is to the  $\tilde{\theta}_U$  steady state.

#### 4.2.2 $\Delta > 0$ in the approach to the steady state

If  $\Delta > 0$ , all mismatched households will sell to flippers in the first instance ( $\alpha = 1$ ), in which case  $n_U = 0$  holds throughout. The dynamics are as described in Section 2, whereby  $(n_F, n_M) \rightarrow (\tilde{n}_F(1), \tilde{n}_M(1))$  and that  $\theta \rightarrow \tilde{\theta}_L$ . There can be no direct convergence to any  $\tilde{\theta}(\alpha)$  steady-state equilibrium, where  $\alpha < 1$ , even if  $\theta$  hits the given  $\tilde{\theta}(\alpha)$  in its approach to  $\tilde{\theta}_L$  since  $\theta = \tilde{\theta}(\alpha)$  is steady state in the system (7) and (11) only for some given  $n_U > 0$ , whereas with  $\alpha = 1$ ,  $n_U = 0$  instead.

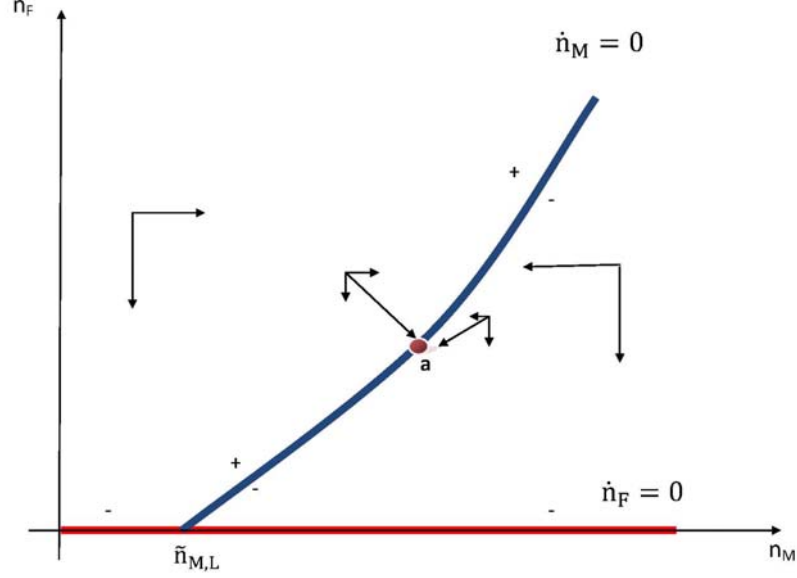


Figure 8: Convergence or divergence to a  $\hat{\theta}$  steady-state equilibrium with  $\alpha = 0$ .

#### 4.2.3 Local Stability of a $\hat{\theta}$ or a $\hat{\theta}_2$ steady-state equilibrium

Consider a  $\hat{\theta}$  steady-state equilibrium, whereby  $\alpha \in (0, 1)$ . Perturb the  $(n_F, n_M)$  pair away from the steady state. Assume that  $\alpha = 0$  because  $\Delta < 0$  off the given steady state. If  $n_F > \tilde{n}_F(\alpha)$  and the deviation of  $n_M$  from  $\tilde{n}_M(\alpha)$  is moderate, the pair will converge back to the given steady state. But if the perturbation results in  $n_F < \tilde{n}_F(\alpha)$  or if  $n_M$  is sufficiently far away from  $\tilde{n}_M(\alpha)$ , the pair will move away from the given steady state and towards the  $\hat{\theta}_U$  steady state. In this case, as  $\theta$  is nearing  $\hat{\theta}_U$ , sooner or later,  $\Delta > 0$  must hold instead. Then  $\alpha$  switches to 1, after which  $n_U$  drops discretely to zero and  $\theta$  moves towards  $\hat{\theta}_L$ . See Figure 9. In the first instance,  $\theta$  will move pass the given  $\hat{\theta}$  since there is a positive  $n_U$  in this steady-state equilibrium but  $n_U = 0$  holds during which  $\theta$  moves pass  $\hat{\theta}$  and towards  $\hat{\theta}_L$ . Meanwhile, when  $\theta$  is sufficiently close to  $\hat{\theta}_L$ ,  $\Delta < 0$  must hold again, triggering  $\alpha$  switching back to 0. That is, the dynamics alternate between those described by Figures 8 and 7, with the  $\dot{n}_F = 0$  locus in Figure 7 for  $\alpha = 1$  and  $b$  the stable  $\hat{\theta}_U$  steady state. Convergence to point  $a$ , the  $\hat{\theta}$  steady-state equilibrium, eventually attains. The same analysis applies to a  $\hat{\theta}_2$  steady-state equilibrium.

#### 4.2.4 Local Stability of a $\hat{\theta}_1$ steady-state equilibrium

Perturb the  $(n_F, n_M)$  pair away from a  $\hat{\theta}_1$  steady-state equilibrium. Assume that  $\alpha = 0$  off the steady state because  $\Delta < 0$ . Just as in the dynamics around a  $\hat{\theta}$  or  $\hat{\theta}_2$  steady-state equilibrium, if  $n_F > \tilde{n}_F(\alpha)$  and if the deviation of  $n_M$  from

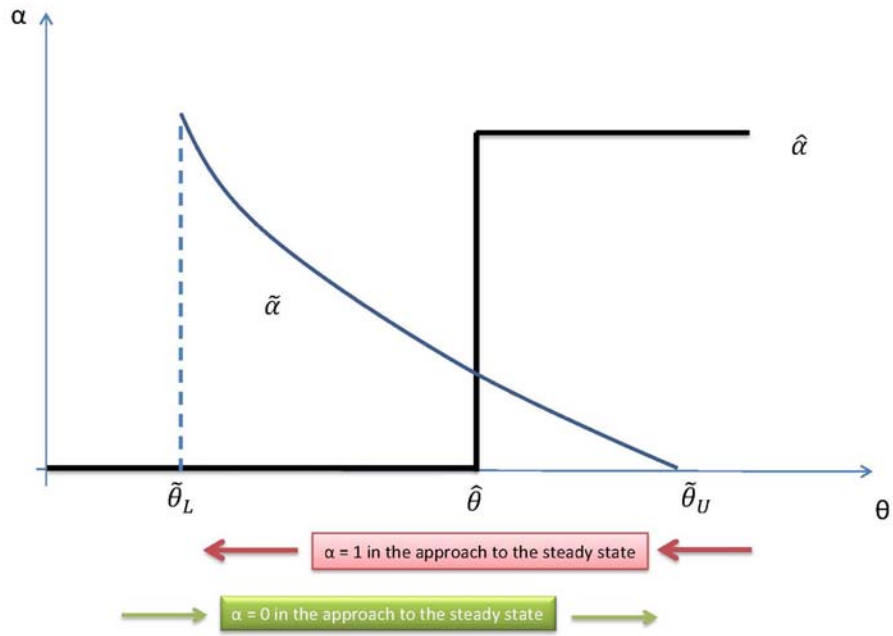


Figure 9: Convergence to a  $\hat{\theta}$  equilibrium

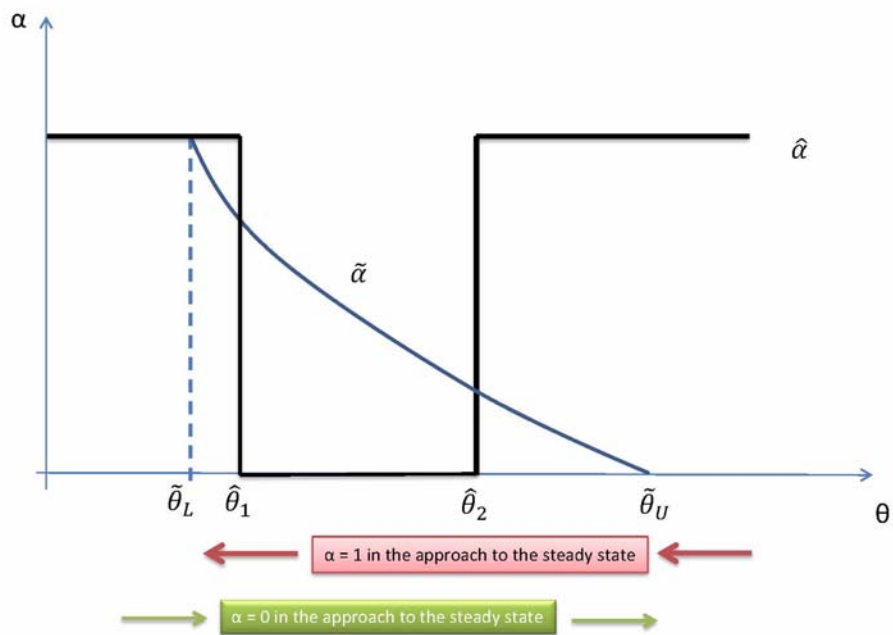


Figure 10: The instability of a  $\hat{\theta}_1$  equilibrium

$\tilde{n}_M(\alpha)$  is moderate, the pair will converge back to the given steady state. But if the perturbation results in  $n_F < \tilde{n}_F(\alpha)$  or if  $n_M$  is sufficiently far away from  $\tilde{n}_M(\alpha)$ , the pair will move away from the given steady state and towards the  $\tilde{\theta}_U$  steady state. In this case, as  $\theta$  moves away from  $\hat{\theta}_1$  and towards  $\tilde{\theta}_U$ ,  $\Delta < 0$ , and therefore  $\alpha = 0$ , should only continue to hold. There will be no convergence back to the initial steady state. See Figure 10. On the other hand, if  $\alpha = 1$  holds off the given steady state because  $\Delta > 0$ , convergence is to the  $\tilde{\theta}_L$  steady state. When  $\theta$  hits the given  $\hat{\theta}_1$  in the first instance, if at all, it will move past it with  $n_U = 0$ . Thereafter, since  $\alpha = 1$  should continue to hold to the left side of  $\hat{\theta}_1$ ,  $\theta$  will converge to  $\tilde{\theta}_L$ . In all, a  $\hat{\theta}_1$  steady-state equilibrium is almost always unstable. We cannot rule out the possibility that  $\alpha = 0$  holds just off the steady state amidst  $n_F > \tilde{n}_F(\alpha)$  and a moderate deviation of  $n_M$  from  $\tilde{n}_M(\alpha)$ , so that the  $(n_F, n_M)$  pair converges directly to the given steady state. But *direct* convergence should be the rare exception rather than the rule. Furthermore, where  $n_F > \tilde{n}_F(\alpha)$ ,  $\theta$  tends to be below  $\hat{\theta}_1$ , in which case  $\Delta < 0$  is unlikely to hold in the first place. In any case, probably only when the initial  $(n_F, n_M)$  pair lies within a tiny subset of the  $(n_F, n_M)$  space can the pair ever converge to a  $\hat{\theta}_1$  steady-state equilibrium. In general, any  $\hat{\theta}_1$  steady-state equilibrium is unlikely to be stable.

We summarize the foregoing analysis as in:

**Proposition 3** *Any dynamic equilibrium in which  $\Delta = 0$  throughout is generally divergent. Both the  $\hat{\theta}$  and  $\hat{\theta}_2$  steady-state equilibria are locally stable. Convergence is usually oscillatory, with  $\Delta$  and  $\theta$  switching signs numerous times before the given steady state obtains. The  $\hat{\theta}_1$  steady-state equilibrium is in general unstable.*

## 5 Concluding remarks

Our stability analyzes are local stability analyzes, and as such, they do not rule out the possibility that there exist multiple equilibrium time paths, each of which approaches a different steady-state equilibrium, for a given initial  $(n_F, n_M)$  pair. We do not know how to demonstrate the possibility analytically. It is not even clear how to approach the problem using numerical methods. We also have not tried analyzing the dynamics in price, vacancy, and transaction around a  $\Delta = 0$  steady-state equilibrium. Where the convergence in  $\theta$  can oscillate, the same certainly should hold for the former variables as well, given that these variables are in the main determined by market tightness.