Inter-Dealer Trades in OTC Markets – Who Buys and Who Sells?

Chung-Yi Tse* and Yujing Xu*

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Abstract

In an OTC market where dealers’ inventory capacities differ, dealers trade among themselves to rebalance inventories for facilitating the sale and purchase of the asset to and from their investor clients. In a market where the asset is sold quickly, the small-capacity dealers sell to the large-capacity dealers to help them replenish their inventories. Conversely, in a slow market where it takes a relatively long time for the asset to be sold, the small-capacity dealers buy from the large-capacity dealers to help them free up inventory capacities. The prediction, though counterintuitive, is supported by some available empirical evidence.

Keywords: OTC Market, Inter-Dealer Trades, Dealers’ Inventories

JEL classifications: D53, D85, G23

*Faculty of Business and Economics, University of Hong Kong. E-mail: cytse@hku.hk (Tse); yujingxu@hku.hk (Xu). We would like to thank the editor and an associate editor of the journal and two referees for very helpful comments and suggestions. Comments from Charles Leung, Dongkyu Chang, Kim-Sau Chung, Pei-yu Melody Lo, the seminar audiences of the City University of Hong Kong, Hong Kong Baptist University and the 2017 Econometric Society Asian meeting also help improve the paper. Xu gratefully acknowledges financial support from HK GRF grant 17517816.
1 Introduction

Many financial assets, including government and corporate bonds, asset-backed securities, and derivatives, are traded in over-the-counter (OTC) markets instead of in centralized exchanges. Two distinguishing features of OTC markets are that the trades are almost always intermediated by dealers of various kinds and that the dealers do not just trade with investors but also among themselves. Indeed, inter-dealer trades can account for a significant fraction of the overall transactions for a given asset.\(^1\)

An important question on inter-dealer trades in OTC markets that seems to have attracted scant attention is how market conditions help shape the directions of trade among heterogeneous dealers.

One instance suggestive of how the trading directions among dealers are influenced by market conditions can be found in Adrian, Fleming, Shachar and Vogt (2017). Specifically, they find that while large dealers expanded their balance sheets much more than small dealers did in the run-up to the 2008 financial crisis, a few small dealers actually expanded their balance sheets during which large dealers were downsizing theirs post-crisis.\(^2\) This difference in balance sheet movements between large and small dealers before and after the 2008 financial crisis could be, to a certain extent, due to large dealers, on the whole, buying from in the up market but selling to small dealers in the down market as equilibrium responses to the changing market environment.

In this paper, we extend the seminal random search models of the OTC market of Duffie, Gårleanu and Pedersen (2005) and Lagos and Rocheteau (2009) to study how dealers trade with one another for managing inventory levels for trading with their customers and how the directions of trade among dealers are determined. The point of departure is that, in our model, dealers are heterogeneous in their inventory capacities. The heterogeneity can be due to risk management considerations, portfolio choices, or can result from differences in financing costs.

In particular, in our model, there is a given measure of what we call small dealers, each

\(^1\)Li and Schürhoff (2019) show that in the period covered by their data set, 20.2 million out of 72.2 million transactions in municipal bonds are inter-dealer trades. A similar percentage of inter-dealer trade is also documented in Hollifield, Neklyudov and Spatt (2017).

\(^2\)As can be seen in Figure 3 of the paper.
endowed with one unit of inventory capacity, and a given measure of what we call large dealers, each endowed with two units of inventory capacity. In each period, investors buy from and sell to dealers in an OTC market in which only dealers who are holding at least a unit of the asset in inventory can sell to and only dealers having at least one unit of spare inventory capacity can buy from investors. Once the investor-dealer trades are completed, and only then, a perfectly competitive inter-dealer market opens, through which dealers trade to rebalance their inventory holdings.

Underlying most of the results in the paper is a particular ranking of the marginal benefits of inventory according to which the first unit of inventory is valued higher by a large dealer than by a small dealer, whereas the small dealer values his last and only possible unit more than the large dealer values his last and second unit. To both the small and the large dealers, having the first unit allows the dealer to sell to investors in the next round of trading. To the small, but not the large, dealer, this is at the expense of exhausting one’s inventory capacity after which the dealer can no longer buy from any investors. The large dealer similarly would forgo any opportunity to buy from investors when he exhausts his inventory capacity in acquiring his second and last unit of the asset but with no compensating benefit since he already has a unit for sale to investors without adding the second unit to inventory.

In other words, small dealers value a unit of the asset in between large dealers starting out with an empty inventory and large dealers starting out with one unit of inventory. The competitive inter-dealer market in equilibrium would then first allocate the asset to large dealers, then to small dealers if there remain units of the asset yet to be allocated to dealers, and finally to large dealers already holding a unit in inventory in case the asset supply is sufficiently large for all small dealers to already hold a unit in inventory. Such allocations are by means of small dealers selling to large dealers in a market with a small asset supply and small dealers buying from large dealers in a market with a large asset supply. Small dealers then should trade with large dealers only but not among themselves while trades between the two types of dealers tend to flow in one direction only in a given market. In our model then, the direction of trade between a small and a large dealers in a given market is persistent, consistent with the finding in Li and Schürhoff (2019).
A dealer is said to provide immediacy for another dealer if the first dealer sells to (buys from) the second dealer when it takes a long time on average for the second dealer to buy (sell) the asset in the market. In our model, large dealers sell to small dealers when there is a large asset supply or when there are only few dealers in the market buying from investors, during which it should be easy for small dealers themselves to buy the asset from investors. When there is a small asset supply or when there are dealers aplenty competing to buy from investors, at which times it should be hard for an individual dealer to buy from investors, it is small dealers who sell to large dealers. Hence, it is the small dealers who trade to provide immediacy for the large dealers in our model, selling to (buying from) large dealers at times during which it takes a long time for the latter to buy (sell) in the market.

The last implication is consistent with the finding in Adrian et al. (2017) if the booming market pre-crisis is time during which it is relatively easy and the market bust post-crisis relatively hard for dealers to find investor buyers in the market. In the earlier period, large dealers gain inventory from small dealers and expand their balance sheets faster. In the later period, small dealers amass inventory from large dealers and expand their balance sheets relative to those of large dealers.

In many OTC markets, dealers can be distinguished by their degrees of centrality, where more central dealers are those who trade with a greater number of dealers than less central ones (Li and Schürhoff (2019) and Hollifield, Neklyudov and Spatt (2017)). The distinction resembles a core-periphery trading structure in the abstract, where the more central dealers can be thought of as core dealers and the less central ones as peripheral dealers. Given that small dealers in our model on the whole trade only with the large dealers, rather than among themselves, whereas the large dealers trade with all dealers, the two types of dealers behave similarly as the less and more central dealers do, respectively, identified in the empirical studies, with regard to the set of dealers they trade with. Under this interpretation, in our model, it is the less central dealers who provide immediacy for the more central dealers. The prediction, counterintuitive as it seems, suggests an explanation for how those classified as peripheral dealers earn a higher markup when they are the last links of the intermediation chain than when they are the first links, reported in Hollifield, Neklyudov and Spatt (2015) and how the
spread between the prices investors pay and receive is higher when they trade with central dealers, reported in Li and Schürhoff (2019).

The equilibrium in our model is constrained efficient in that the allocation of inventories and spare capacities among dealers falling out from inter-dealer trades in equilibrium coincides with the planning optimum. That the two allocations coincide perhaps is not surprising given that dealers trade an indivisible asset in a competitive market. More interestingly, it suggests that for efficiency, small peripheral dealers indeed should trade to provide immediacy for large central dealers.

A further novel result in our model is that the inter-dealer trading volume and the asset supply relation is “M-shaped”. Dealers trade among themselves to rebalance inventory, to which the need is greatest when they find it hardest to acquire inventory or liquidity from investors; i.e., when the asset supply is at the lowest or highest level. But precisely when the asset supply is at the lowest or highest level, dealers who possess inventory to sell or spare capacity to buy can only be few and far between. Increases in the asset supply from lowest level and decreases from the highest level should then give rise to more trades. For intermediate levels of asset supply, dealers buy from and sell to investors both with relative ease, largely alleviating any need for rebalancing inventory. In this way, the trading volume peaks at two levels of asset supply – when it is moderately low and when it is moderately high. In contrast, in one extension in which we consider a frictional inter-dealer market, the trading volume-asset supply relation is bell-shaped, with the volume peaking at an intermediate asset supply.3 From this difference is a readily implementable empirical test of the efficiency of a given inter-dealer market.

Related Literature  Our framework is adapted from the seminal models of OTC markets in Duffie et al. (2005) and Lagos and Rocheteau (2009). In these models, to simplify, the authors assume that whenever a dealer trades with an investor, the dealer can instantaneously offset the transaction by trading in a perfectly competitive inter-dealer market that opens at all times. Such an environment, in which a dealer trades with another dealer only if and when

3Hugonnier, Lester and Weill (2018) and Kiyotaki and Wright (1989) find the same bell-shaped relation in their models of frictional market.
he meets an investor, is apparently not set up to study inter-dealer trades as the trades cannot possibly exhibit any distinctive structure.

In this paper, we extend the two aforementioned models by assuming that dealers only have periodic access to the inter-dealer market, giving rise to a model in which dealers may choose to hold inventory to facilitate future trades. Dealers in Lagos, Rocheteau and Weill (2011) and Weill (2011), respectively, also have incentives to hold inventory when there is a negative shock knocking the market off the steady state and when there is a transient selling pressure in a competitive dynamic market.

Our primary contribution is an investigation of how the directions of trades among dealers facing different inventory constraints are determined and the implications thereof on how small peripheral dealers provide immediacy for large central dealers. Our paper then contributes to the literature on how dealers specialize and form a core-periphery trading network in which the trading direction is also persistent. The literature has studied how a dealer becomes a core dealer when his search ability is high (Neklyudov (2015)), when he invests more to raise his contact rate (Farboodi, Jarosch and Shimer (2018)) or to improve his bargaining skills (Farboodi, Jarosch, Menzio and Wiriadinata (2018)), or when the dealer specializes in serving clients who trade frequently (Sambalaibat (2018)), how the usual network externality should lead to most dealers choosing to set up costly connections with just a handful of dealers (Wang (2017)), and how information imperfection should result in individual agents specializing in market making and taking on the role of core dealers (Chang and Zhang (2019)).

Dealers’ inventory constraints become especially relevant in the aftermath of the 2008 financial crisis. Our paper adds to the growing literature on the roles of such constraints in determining the directions, structure and volume of inter-dealer trades. In particular, Dunne, Hau and Moore (2015) study the effects of the interaction of inventory constraint and adverse selection on market stability while Cimon and Garriott (2018) show how dealers are adapting to the recent regulations on inventory by shifting to an agency basis of trade. Besides, Choi and

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4 Other explanations for why dealers trade among themselves in the literature include the amassment of cash endowment in Colliard and Demange (2017), the mitigation of information asymmetry in Glode and Opp (2016), and the sharing of inventory risks among risk averse dealers in Ho and Stoll (1983), Atkeson, Eisfeldt and Weill (2015) and Üslü (2019).
Huh (2018) find that customers, not dealers, are increasingly the liquidity providers post-crisis, a finding in line with our prediction that small dealers provide immediacy for large ones.

Small dealers in our model trade to provide immediacy for large dealers as they value a unit of inventory in between large dealers having an empty inventory and large dealers already holding a unit inventory. There is a subtle similarity to how investors having an intermediate valuation of the asset in Hugonnier et al. (2018) and Shen, Wei and Yan (2018) endogenously become dealers, buying from investors with the lowest valuation and then staying on the market selling to investors with the highest valuation. The difference between our model and theirs is that the difference in valuation in ours arises endogenously and that the small dealers in our model, as intermediaries for large dealers, either just sell to or buy from their large dealer customers in a given market.

The rest of the paper is organized as follows. In Section 2, we set up the model and then study the model’s equilibrium. We discuss the model’s implications on trading directions between small and large dealers and compare those implications against the available empirical evidence in Section 3. In Section 4, we explore two additional implications of the model as pertaining to how dealers’ inventories and the inter-dealer trading volume vary with changes in the market environment. Section 5 shows how the equilibrium allocation coincides with the planning optimum. In Section 6, we discuss two extensions of the model and demonstrate how the major results hold in more general settings. Section 7 concludes. All proofs are relegated to the Appendix.

2 Model and Analysis

2.1 Basic Environment

Time is discrete and runs forever. Two groups of agents – investors and dealers – buy and sell an asset with supply fixed at $A$ in an OTC market. An investor can hold either zero or one unit of the asset at a time. A high-valuation investor derives a per period return $\nu > 0$ in holding a unit of the asset, whereas low-valuation investors and dealers derive the same return

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5 A similar mechanism is at work in Piazzesi and Schneider (2009) in their analysis of the housing market.
normalized to zero. Investors can only buy and sell the asset through dealers of which there are two types: (1) small dealers, each of whom can hold up to one unit of the asset at a time and (2) large dealers, each of whom can hold up to two units. There is a fixed measure $n^{SD}$ of small dealers and a fixed measure of $n^{LD}$ of large dealers in the market. All agents are risk neutral and discount the future at the same factor $\beta$.

At the beginning of each period, a measure of $e$ investors enter the market as high-valuation investors with no assets in hand. Together with the entrants in previous periods who remain as high-valuation investors but have yet to acquire a unit of the asset, they constitute the population of investor-buyers ($I_B$) in the market, whose measure is denoted as $n^I_B$. The low-valuation investors who own a unit of the asset become the investor-sellers ($I_S$) in equilibrium, whose measure is denoted as $n^I_S$. At the end of the period, each high-valuation investor is hit by a liquidity shock at probability $\delta \in (0, 1)$ and turns into a low-valuation investor forever thereafter. Those who own a unit of the asset turn into investor-sellers in the next period and those who do not cease to remain as investor-buyers but simply exit the market.

Each period is divided into two subperiods. In the first subperiod, a decentralized investor-dealer market opens in which the bilateral meetings between investors and dealers take place as governed by a constant-returns matching function. Define market tightness

$$\theta = \frac{n^D}{n^I_B + n^I_S},$$

as the ratio of the measure of all dealers $n^D = n^{SD} + n^{LD}$ to the measure of all active investors on the market. Each investor is randomly matched with a dealer at probability $\eta(\theta) \in [0, 1]$, whereas a dealer is randomly matched with an investor at probability $\mu(\theta) = \eta(\theta)/\theta$. The meeting probability $\eta(\theta)$ satisfies the usual conditions:

$$\frac{\partial \eta}{\partial \theta} > 0; \quad \frac{\partial^2 \eta}{\partial \theta^2} < 0; \quad \lim_{\theta \to 0} \frac{\partial \eta}{\partial \theta} = 1; \quad \lim_{\theta \to \infty} \frac{\partial \eta}{\partial \theta} = 0.$$

Only dealers holding at least a unit of the asset in inventory can sell to the investor-buyers and only dealers having at least a unit of spare inventory capacity can buy from the investor-sellers that they meet. Prices in the investor-dealer market fall out of the bargaining between

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\*As in other search-based models of OTC market that use an overlapping-generation structure (Vayanos and Wang (2007), Vayanos and Weill (2008), Afonso (2011) and Sambalaibat (2018)), all investors, irrespective of ownership status, are hit by the liquidity shock at the same probability at the end of each period.
the buyers and sellers in the bilateral meetings in which the agents on the two sides split
the surplus in equal halves. Those investor-buyers who succeed in buying a unit would start
collecting the payoff $v$ in the next period as long as they remain as high-valuation investors,
whereas those investor-sellers who succeed in selling their units leave the market for good.

In the second subperiod, a competitive inter-dealer market opens in which any and all
dealers buy and sell as many units of the asset among themselves as they see fit at a given
market price.\(^7\) The analysis of how dealers trade among themselves in the market to rebalance
inventories to prepare for the next round of trading with their investor customers is the main
focus of the paper.

For brevity, we restrict attention to studying steady-state equilibria in this paper.

### 2.2 Value Functions

A small dealer, $S_i$, $i = 0, 1$, is either holding 0 or 1 unit of the asset at a time. Write $V^SD_i$ as
the value function of an $S_i$ entering the investor-dealer market in the first subperiod with an
$i$-unit inventory and $W^SD_i$ the value function of the dealer entering the inter-dealer market in
the second subperiod. If the asset is traded in the inter-dealer market at price $p$,

\[
W^SD_0 = \max \left\{ \beta V^SD_0, \beta V^SD_1 - p \right\}, \tag{2}
\]

\(^7\)We assume that all dealers, regardless of their inventory capacities, trade in the inter-dealer market on
equal ground. The assumption seemingly contradicts the fact that in selected OTC derivatives markets, there
are clearinghouses in which members and only members may clear their trades centrally. We emphasize that
the clearinghouses are not quite inter-dealer markets. Dealers find trading partners first in the inter-dealer
market and then may clear their trades centrally in the clearinghouses. Trade clearing, in particular, refers
to the settlement of a trade after its agreement by the counterparties concerned, which for certain derivatives,
may take decades. The advantages of centralized over bilateral clearing include multilateral netting and trade
compressions, both serving to help reduce the amount of collateral dealers may have to put up for trading.
While only the largest banks and financial institutions are members of the major clearinghouses, non-members
may clear their trades in clearinghouses by paying members to do so on their behalf. See Baker (2016) and
Chang (2016) for details. Large dealers, perhaps by virtue of their membership of the major clearinghouses, may
trade at a lower cost, which is possibly one source for their larger inventory capacity. In any case, empirically,
there exists no evidence to suggest that some dealers have better access to the inter-dealer market for seeking
out trading partners than others.
\[ W_i^{SD} = W_0^{SD} + p. \tag{3} \]

A large dealer \( L_i, i = 0, 1, 2 \), can hold up to two units of the asset in inventory. Then, for \( V_i^{LD} \) denoting the value function of a large dealer entering the investor-dealer market, the value function of the dealer entering the inter-dealer market is

\[ W_0^{LD} = \max \left\{ \beta V_0^{LD}, \beta V_1^{LD} - p, \beta V_2^{LD} - 2p \right\}, \tag{4} \]

\[ W_i^{LD} = W_0^{LD} + ip, \text{ for } i = 1, 2. \tag{5} \]

In (2), an \( S_0 \) entering the inter-dealer market chooses between buying a unit and not buying. In (4), an \( L_0 \) entering the inter-dealer market may choose to buy up to two units. Eqs. (3) and (5) follow as a unit of the asset is worth \( p \) in the competitive inter-dealer market.

We call any dealer having at least a unit of spare inventory capacity to buy from investors a dealer-buyer. A dealer-buyer gains from buying from an investor for the amount \( p \), for what a unit of the asset is worth in the inter-dealer market, minus the payment the dealer makes to the investor for the unit. To the investor-seller, the gain from trade is equal to the payment from the dealer net of the (discounted) continuation value of being an investor-seller \( (U^I_S) \). The surplus of trade between an investor-seller and any dealer-buyer, because of competition in the inter-dealer market, is then simply equal to

\[ z_{IS} = \max \left\{ p - \beta U^I_S, 0 \right\}. \tag{6} \]

Dealers having an empty inventory (\( S_0 \)s and \( L_0 \)s), who may buy from but not sell to investors, are dealer-buyers only. Then, since a dealer meets a randomly selected investor at probability \( \mu (\theta) \), among whom a fraction \( \frac{n^I_S}{n^I_S + n^I_B} \) are sellers, the respective value functions of an \( S_0 \) and an \( L_0 \) entering the investor-dealer market are equal to

\[ V_0^{SD} = W_0^{SD} + \mu (\theta) \frac{n^I_S}{n^I_S + n^I_B} z_{IS}, \tag{7} \]

\[ V_0^{LD} = W_0^{LD} + \mu (\theta) \frac{n^I_S}{n^I_S + n^I_B} z_{IS}. \tag{8} \]

Let \( n_i^{SD} \) and \( n_i^{LD} \) denote the respective measures of small and large dealers holding an \( i \)-unit inventory in the investor-dealer market. The measure of all dealer-buyers is thus,

\[ n_B^D = n_0^{SD} + n_0^{LD} + n_1^{LD}. \]
Then, on the other side of the market, since an investor meets a randomly selected dealer at probability $\eta(\theta)$, among whom a fraction $\frac{n_D^I}{n_D^I + n_D^S}$ are dealer-buyers, the value function of an investor-seller is equal to

$$U^I_S = \eta(\theta) \frac{n_D^I}{n_D^I + n_D^S} z_{IS} + \beta U^I_S.$$  \hfill (9)

We call any dealer who holds at least a unit of the asset in inventory for sale to investors a dealer-seller. A dealer-seller gains from selling to an investor for the payment he receives from the investor net of the value of the asset $p$ in the inter-dealer market. To the investor-buyer, the gain from trade is equal to the capital gain of acquiring the unit minus the payment made to the dealer. The investor-buyer’s capital gain, for $U^Q_H$ and $U^I_B$ denoting, respectively, the value of a high-valuation investor holding a unit of the asset (high-valuation owner hereafter) and the value of an investor-buyer, is equal to $(1 - \delta) U^Q_H + \delta U^I_B - (1 - \delta) U^I_B$ given that the investor only remains a high-valuation investor in the next period absent any liquidity shock. All this means that the surplus of trade between an investor-buyer and any dealer-seller, with a competitive inter-dealer market, is equal to

$$z_{IB} = \max \left\{ \beta \left( (1 - \delta) \left( U^Q_H - U^I_B \right) + \delta U^I_B \right) - p, 0 \right\}. \hfill (10)$$

Dealers holding a full inventory ($S_1$s and $L_2$s), who may sell to but not buy from investors, are dealer-sellers only. Then, the respective value functions of an $S_1$ and an $L_2$ entering the investor-dealer market are equal to

$$V^{SD}_1 = W^{SD}_1 + \mu(\theta) \frac{n_I^B}{n_S^I + n_B^I} \frac{z_{IB}}{2}, \hfill (11)$$

$$V^{LD}_2 = W^{LD}_2 + \mu(\theta) \frac{n_I^B}{n_S^I + n_B^I} \frac{z_{IB}}{2}. \hfill (12)$$

On the other side of the market, the value function of an investor-buyer is equal to

$$U^I_B = \eta(\theta) \frac{n_S^D}{n_D^I} z_{IB} + (1 - \delta) \beta U^I_B, \hfill (13)$$

where

$$n_S^D = n_1^{SD} + n_1^{LD} + n_2^{LD},$$

is the measure of all dealer-sellers.
An $L_1$, in holding a unit inventory and having a unit of spare inventory capacity, is both a dealer-buyer and a dealer-seller, who can trade whether the investor he happens to meet is an investor-buyer or investor-seller. Then, the value function of the dealer is equal to

$$V_{LD}^1 = W_{LD}^1 + \mu(\theta) \left( \frac{n_S^1}{n_S^1 + n_B^1} z_{IS} + \frac{n_B^1}{n_S^1 + n_B^1} z_{IB} \right).$$

(14)

Finally, a high-valuation owner derives a per period return $\upsilon$ from holding a unit of the asset and turns into a low-valuation investor cum investor-seller at probability $\delta$ at the end of the period in which case,

$$U_{ON}^H = \upsilon + \beta \left( (1 - \delta) U_{ON}^H + \delta U_{IS}^I \right).$$

(15)

In closing, we should mention that there must be the same price $p_{IS} = p - z_{IS}/2$ an investor-seller receives whoever dealers he is selling to given that there is same surplus $z_{IS}$ is any such trades. By the same token, there is the same price $p_{IB} = p + z_{IB}/2$ an investor-buyer pays whoever dealers he is buying from.

### 2.3 Inter-dealer Market Trades

By (2) and (3), whether an $S_0$ entering the inter-dealer finds it optimal to buy and whether an $S_1$ finds it optimal to sell depend on how the inter-dealer market price $p$ compares with $\beta \left( V_{SD}^1 - V_{SD}^0 \right)$. Similarly, large dealers entering the market make their buy and sell decisions by comparing $p$ against $\beta \left( V_{LD}^1 - V_{LD}^0 \right)$ and $\beta \left( V_{LD}^2 - V_{LD}^1 \right)$.

**Proposition 1** $V_{LD}^1 - V_{LD}^0 \geq V_{SD}^1 - V_{SD}^0 \geq V_{LD}^2 - V_{LD}^1$. The first inequality is strict if and only if $z_{IS} > 0$ whereas the second inequality is strict if and only if $z_{IB} > 0$.

Proposition 1 says that an $L_0$ has the most to gain from acquiring a unit of the asset in the inter-dealer market, followed by an $S_0$, whereas an $L_1$ has the least to gain. In our model, all dealers may each meet up to one investor, who may be a seller or a buyer, in a given round of trading. Both large dealers and small dealers holding a unit inventory would be able to trade when meeting an investor-buyer. The additional gain a large dealer enjoys from holding a unit inventory versus a small dealer is flexibility: an $L_1$, but not an $S_1$, can also trade when
meeting an investor-seller. This explains the first inequality of the Proposition. If a dealer earns a strictly positive surplus from buying from an investor where $z_{IS} > 0$, the inequality is strict. There is the same cost of holding a full inventory for the two types of dealers in forgoing any opportunity to buy from an investor in the next round of trading. The small dealer, however, benefits from acquiring the unit as he may then be able to trade with an investor-buyer should he meet one in the next period. There is not any such upside to the purchase by a large dealer already holding a one-unit inventory as the dealer can sell without adding a unit to inventory. This explains the second inequality of the Proposition. If a dealer earns a strictly positive surplus from selling to an investor where $z_{IB} > 0$, the inequality is strict.

Who buys and who sells in the inter-dealer market depend on what price clears the market, a price that must be bounded by

$$p \in \left[ \beta \left( V_{2}^{LD} - V_{1}^{LD} \right), \beta \left( V_{1}^{LD} - V_{0}^{LD} \right) \right]$$

in equilibrium since at any $p$ above the upper bound of the interval, there can only be sellers and at any $p$ below the lower bound, there can only be buyers in the market. Besides, for any $p$ not exactly equal to $\beta$ times one of the three marginal benefits in Proposition 1, any and all dealers who desire to trade either strictly prefer to buy or sell. In this case, the market clears only if the parameters conspire to just equate the measures of buyers and sellers. But such a parameter configuration can at best make up a zero-measure subset of the parameter space.\(^8\) For $p$ equal to $\beta$ times one of the three marginal benefits, there is one type of dealer holding a given inventory indifferent between selling and not selling or between buying and not buying. The market may then clear at some mixing probability for the mixed strategy played by the marginal buyers or sellers.

**Proposition 2** For $p = \beta \left( V_{1}^{LD} - V_{0}^{LD} \right)$ or $\beta \left( V_{1}^{SD} - V_{0}^{SD} \right)$, both $z_{IS}$ and $z_{IB}$, and $p$ itself are strictly positive, whereas for $p = \beta \left( V_{2}^{LD} - V_{1}^{LD} \right)$, $z_{IS}$ and $p$ itself are equal to zero while $z_{IB} > 0$.

\(^8\)See the Online Appendix for the formal proof.
The case for \( p = \beta (V_2^{LD} - V_1^{LD}) \) deserves further explanation. When an \( L_1 \) turns into an \( L_2 \), the dealer can no longer be a dealer-buyer, whereas the dealer is a dealer-seller with or without adding a unit to his inventory. The dealer must then be worse off buying the unit at any positive \( p \) and is at best indifferent at \( p = 0 \) and that dealers do not gain from buying from investors at all where \( z_{I_0} = 0 \).\(^9\) There must be a positive surplus in an investor-buyer trade though \( (z_{I_1} > 0) \) since the investor, but not the dealer, is better off owning a unit than otherwise.

It is useful to classify equilibrium into three types, corresponding to \( p \) equal to each candidate equilibrium price.

The “Selling” Equilibrium  In the Selling Equilibrium, \( p = \beta (V_1^{LD} - V_0^{LD}) \). By Propositions 1 and 2,

\[
p = \beta (V_1^{LD} - V_0^{LD}) > \beta (V_i^{SD} - V_0^{SD}) > \beta (V_2^{LD} - V_1^{LD}),
\]

in which case no dealers strictly prefer to buy with \( p \) anchored at the highest possible marginal value of inventory. In the meantime, any dealers with a full inventory strictly prefer to sell. For this reason, we call this the Selling Equilibrium in which the optimal inventory of a small dealer is zero unit whereas that of a large dealer is zero or one unit. For the market to clear, a fraction or all of \( L_0s \) must buy since they are the only possible buyers. And if there are sufficiently many \( L_0s \) to meet the supply out of all \( S_1s \) and \( L_2s \) each selling one unit, the market can clear. Specifically, let \( m_i^{SD}, i = 0, 1 \) and \( m_i^{LD}, i = 0, 1, 2 \), be the respective measures of small and large dealers entering the inter-dealer market holding an \( i \)-unit inventory. The inter-dealer market can clear at \( p = \beta (V_1^{LD} - V_0^{LD}) \) for

\[
m_0^{LD} \geq m_1^{SD} + m_2^{LD}. \tag{16}
\]

Because each \( L_1 \) is indifferent between selling and not selling, if (16) holds as a strict inequality, there is room for a fraction or all of them selling in equilibrium. In this way, there

\(^9\)A \( p = 0 \) results from the normalization that both low-valuation investors and dealers derive zero flow payoff from owning the asset. With a positive normalized payoff, \( p \) would become positive without affecting the qualitative results to follow.
is a continuum of equilibrium, indexed by the measure of \( L_1 \) sellers. In equilibrium, since a sale by an \( L_1 \), who will become an \( L_0 \) afterward, must be matched by a purchase by an \( L_0 \), who will become an \( L_1 \) afterward, such trades merely result in those agents switching identities. More importantly, the multiplicity has no bearing at all on the conditions for the existence of the equilibrium and the allocation that follows as they are derived solely from the optimal inventories of dealers. The same kind of multiplicity in the next two types of equilibrium we define in the following is similarly inconsequential.

**The “Balanced” Equilibrium** In the Balanced Equilibrium, \( p = \beta (V_1^{SD} - V_0^{SD}) \). By Propositions 1 and 2,

\[
\beta (V_1^{LD} - V_0^{LD}) > p = \beta (V_1^{SD} - V_0^{SD}) > \beta (V_2^{LD} - V_1^{LD}), \]

from which it follows that \( L_0 \)s strictly prefer to buy one unit while \( L_2 \)s strictly prefer to sell one unit. We refer to this as the Balanced Equilibrium, in which the optimal inventory of a large dealer is one unit, whereas that of a small dealer is zero or one unit. For the inter-dealer market to clear, if large dealers buying (selling) outnumber large dealers selling (buying) in the market, small dealers on balance must sell (buy). That is, in case \( m_0^{LD} \geq m_2^{LD} \), the Balanced Equilibrium obtains for

\[
m_1^{SD} \geq m_0^{LD} - m_2^{LD}. \tag{17} \]

Otherwise \( m_2^{LD} \geq m_0^{LD} \), the market can clear at the given \( p \) for

\[
m_0^{SD} \geq m_2^{LD} - m_0^{LD}. \tag{18} \]

It is useful to refer to the first case as the Balanced-Selling Equilibrium and the second case as the Balanced-Buying Equilibrium.

**The “Buying” Equilibrium** In the Buying Equilibrium, \( p = \beta (V_2^{LD} - V_1^{LD}) \). By Propositions 1 and 2,

\[
\beta (V_1^{LD} - V_0^{LD}) = \beta (V_1^{SD} - V_0^{SD}) > \beta (V_2^{LD} - V_1^{LD}) = p = 0, \]

from which it follows that no dealers strictly prefer to sell with \( p \) anchored at the lowest possible marginal value of inventory. Meanwhile, any dealers with an empty inventory strictly prefer to
buy. For this reason, we call this the Buying Equilibrium in which the optimal inventory of a large dealer is one or two units, whereas that of a small dealer is one unit. For the inter-dealer market to clear at the given \( p \), there should be sufficiently many \( L_2 \)s selling in the market to meet the demand out of all \( S_0 \)s and \( L_0 \)s each buying one unit; i.e.,

\[
m_2^{LD} \geq m_0^{SD} + m_0^{LD}.
\]

(19)

In Table 1, we summarize the identities of the buyers and sellers upon entry into and the optimal inventories of large and small dealers upon exiting the inter-dealer market. Notice that in all three equilibrium types, at least a fraction of \( L_0 \)s buy and at least a fraction of \( L_2 \)s sell. The defining difference among the equilibria is the role played by small dealers. In the Selling Equilibrium, \( S_1 \)s sell while \( S_0 \)s stay out of the market. In the Buying Equilibrium, \( S_0 \)s buy while \( S_1 \)s stay out of the market. In the Balanced Equilibrium, small dealers may either sell or buy, depending on whether or not the buyers among large dealers outnumber the sellers.

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Buyers</th>
<th>Sellers</th>
<th>Optimal Inventory</th>
<th>Restrictions on ( n_i^{SD} ) and ( n_i^{LD} )</th>
</tr>
</thead>
</table>
| Selling     | \( L_0 \) | \( L_2^*, S_1^*, (L_1) \) | 0 | 1 and 0 | \( n_0^{SD} = n^{SD} \)
|             |        |         |                   |                                          | \( n_1^{SD} = n_2^{LD} = 0 \) |
| Balanced    | \( L_0^* \), \( S_0 \) | \( L_2^* \) | 0 and 1 | 1 | \( n_1^{LD} = n^{LD} \)
| -Selling    |        |         |                   |                                          | \( n_0^{LD} = n_2^{LD} = 0 \) |
| -Buying     | \( S_0^* \), \( L_0^*, (L_1) \) | \( S_1 \) | 1 | 1 and 2 | \( n_1^{SD} = n^{SD} \)
| Buying      |        | \( L_2 \) |                   |                                          | \( n_0^{SD} = n_0^{LD} = 0 \) |

* inframarginal buyers and sellers who gain from trade; ( ) dealers who are indifferent to trading and do not need to trade for market clearing

Table 1: Characteristics of the three types of equilibrium

Each candidate equilibrium places a set of restrictions on the inventories held by small and large dealers and may obtain only if the inter-dealer market can clear at the given \( p \). We next
proceed to complete the definitions of the equilibria and derive the conditions for each type of equilibrium to hold.

2.4 Equilibrium Conditions

If the market is populated by \( n^{SD} \) small dealers and \( n^{LD} \) large dealers, then

\[
\begin{align*}
n_0^{SD} + n_1^{SD} &= n^{SD}, \quad (20) \\
n_0^{LD} + n_1^{LD} + n_2^{LD} &= n^{LD}. \quad (21)
\end{align*}
\]

The asset is in fixed supply equal to \( A \), and hence,

\[
\begin{align*}
n_{ON}^H + n_{IS} + n_1^{SD} + n_1^{LD} + 2n_2^{LD} &= A, \quad (22)
\end{align*}
\]

where \( n_{ON}^H \) denotes the measure of high-valuation owners.

In the steady state, the respective inflows and outflows of high-valuation owners, investor-sellers, and investor-buyers are equal. That is,

\[
\begin{align*}
(1 - \delta) n_{IB} \eta(\theta) \frac{n_S^D}{n_D} &= \delta n_{ON}^H, \quad (23) \\
\delta \left( n_{ON}^H + n_{IS} + n_1^{SD} \right) \eta(\theta) \frac{n_S^D}{n_D} &= \eta(\theta) \frac{n_B^D}{n_D} n_{IS}, \quad (24) \\
e &= \left( \delta + (1 - \delta) \eta(\theta) \frac{n_S^D}{n_D} \right) n_{BS}. \quad (25)
\end{align*}
\]

Not all \( n_i^{SD} \) and \( n_i^{LD} \) can be positive in a given type of equilibrium. For example, in the Selling Equilibrium, because all small dealers exit the inter-dealer market with an empty inventory, \( n_0^{SD} = n^{SD} \) and \( n_1^{SD} = 0 \) must hold, and because large dealers may do so with either an empty or a one-unit inventory, \( n_2^{LD} = 0 \) must hold. The respective restrictions on \( n_i^{SD} \) and \( n_i^{LD} \) in the three types of equilibrium are as depicted in the last column of Table 1.

Small dealers entering the inter-dealer market in the second subperiod with an empty inventory are among the \( S_0 \)s entering the investor-dealer market in the first subperiod who fail to buy a unit in the market and the \( S_1 \)s who succeed in selling the unit they each possess. This means that

\[
\begin{align*}
n_0^{SD} = \left( 1 - \mu(\theta) \frac{n_S^I}{n_S^I + n_B^I} \right) n_0^{SD} + \mu(\theta) \frac{n_B^I}{n_S^I + n_B^I} n_1^{SD}. \quad (26)
\end{align*}
\]
The equations for $m_i^{SD}$ and $m_i^{LD}$, $i = 0, 1, 2$, are defined similarly and can be found in Appendix 8.1.

### 2.5 Equilibrium

Given $\{n^{SD}, n^{LD}, A, e, \delta\}$, a steady-state equilibrium consists of the respective non-negative values of $n_0^{SD}, n_1^{SD}, n_0^{LD}, n_1^{LD}, n_2^{LD}, n_H^N, n_S^i$ and $n_B^i$ that satisfy (20)-(25), the restrictions on $n_i^{SD}$ and $n_i^{LD}$ in Table 1 and the market-clearing conditions for the type of equilibrium under consideration in (16)-(19), with $m_0^{SD}$ given by (26) and $m_1^{SD}$ and $m_i^{LD}$ given by (31)-(34) in Appendix 8.1. A given steady-state equilibrium, under this definition, may be made up of a continuum indexed by the actual measure of the indifferent marginal sellers or buyers in the inter-dealer market. This multiplicity, as we argue before, is inconsequential as having no effects on the endogenous variables of interest.

**Proposition 3** A unique steady-state equilibrium as defined above exists. For $A \in \left[0, n^{LD} + \frac{e}{\delta}\right)$, define

$$
\Omega_S (A) = \frac{\delta}{1 - \delta} \frac{(A - n^{LD} - \frac{n^{LD} e}{nD \delta})(n^{LD} + nD) - \mu (\frac{n^{SD}}{\delta} + n^{LD} - A nD + n^{LD})}{\frac{l}{n^{LD} n^{SD}}},
$$

where $\Omega_S(0) < 0$, $\partial \Omega_S (A) / \partial A > 0$, and $\lim_{A \rightarrow n^{LD} + \frac{e}{\delta}} \Omega_S(A) > 0$, so that there exists a unique $A_S \in \left(0, n^{LD} + \frac{e}{\delta}\right)$ satisfying $\Omega_S (A_S) = 0$. For $A \in \left(nD + \frac{e}{\delta}, \infty\right)$, define

$$
\Omega_B (A) = \frac{\delta}{1 - \delta} \frac{(nD + \frac{nD e}{nD \delta} - A)(n^{LD} + nD) - \mu (\frac{n^{SD}}{\delta} + n^{LD} - A nD + n^{LD})}{\frac{l}{n^{SD} n^{D}}},
$$

where $\lim_{A \rightarrow nD + \frac{e}{\delta}} \Omega_B (A) > 0$, $\partial \Omega_B (A) / \partial A < 0$ and $\lim_{A \rightarrow \infty} \Omega_B (A) < 0$, so that there exists a unique $A_B \in \left(nD + \frac{e}{\delta}, \infty\right)$ satisfying $\Omega_B (A_B) = 0$.

(a) For $A \leq A_S$, the Selling Equilibrium holds. As $A$ increases up to $A = A_S$, $n_1^{LD} = n^{LD}$.

(b) For $A \in \left(A_S, n^{LD} + \frac{n^{SD} e}{2} + \frac{e}{\delta}\right)$, the Balanced Equilibrium with $n_1^{SD} < n^{SD}/2$ holds. This happens to be the Balanced-Selling Equilibrium, in which small dealers sell to clear the inter-dealer market, as the condition $n_1^{SD} < n^{SD}/2$ is equivalent to $m_0^{LD} > m_2^{LD}$ and also $n_B^i > n_S^i$. As $A$ increases up to $A = n^{LD} + \frac{n^{SD} e}{2} + \frac{e}{\delta}$, $n_1^{SD} = n^{SD}/2$ and $m_0^{LD} = m_2^{LD}$ so that the sales and purchases made by large dealers just clear the inter-dealer market.
Figure 1: Equilibrium – Existence and Uniqueness

(c) For $A \in \left( n^{LD} + \frac{n^{SD}}{2} + \frac{e}{\beta}, A_B \right]$, the Balanced Equilibrium with $n_1^{SD} > n^{SD}/2$ holds. This is also the Balanced-Buying Equilibrium, in which small dealers buy to clear the inter-dealer market. As $A$ increases up to $A = A_B$, $n_1^{SD} = n^{SD}$.

(d) For $A > A_B$, the market turns into the Buying Equilibrium in which $n_2^{LD} > 0$.

Proposition 3 essentially describes how the asset would be allocated to the two types of dealers in equilibrium for different levels of $A$ for a given asset demand as measured by the rate $e$ at which investors enter the market, as illustrated in Figure 1. Part (a) is for when the asset supply is at the lowest level, whereupon there would only be enough of the asset left, after accounting for the amount held by investors, to allow for a fraction of large dealers, who value the first unit of the asset the most, to each hold a unit in inventory. The Selling Equilibrium holds as a result and then it holds up to where the market has enough of the asset for each large dealer to hold a unit, at which point the Balanced Equilibrium comes into play. When the asset supply is still at a relatively low level, as for Part (b), there would only be fewer
than one-half of all small dealers holding a unit inventory. At such a level of asset supply, it turns out that there would be fewer investor-sellers left in the market than -buyers, given the relative ease with which investors can sell in a market with a scarce supply. If there are more investor-buyers than -sellers, more of the large dealers, who all enter the investor-dealer market as $L_1$s, would manage to sell than to buy. Since those who sell turn into $L_0$ buyers and those who buy turn into $L_2$ sellers in the inter-dealer market afterwards, there would be more buyers than sellers among large dealers entering the market. Small dealers then sell to eliminate the excess demand among large dealers. Part (c) of the Proposition is for $A$ large enough to result in more than one-half of all small dealers to hold a unit in inventory, giving rise to an equilibrium in which small dealers buy to eliminate the excess supply among large dealers. When the asset supply has risen to the level to allow for all small dealers to each hold a unit in inventory, the Balanced Equilibrium eventually gives way to the Buying Equilibrium, as referred to by Part (d) of the Proposition, in which a fraction of large dealers would hold a two-unit inventory.

It turns out that the equations in (26) and (31)-(34) in Appendix 8.1 linking the measures of dealers in the inter-dealer to the investor-dealer markets, together with the restrictions on $n_{SD}^i$ and $n_{LD}^i$ in Table 1, suffice to guarantee that the market-clearing conditions in (16)-(19) would be satisfied. Recall that all dealers exit the inter-dealer market and enter the investor-dealer market with their respective optimal inventories. Any dealers that trade with investors afterwards would weakly prefer to trade in the inter-dealer market again to restore their respective optimal inventories. Those dealers who strictly prefer to trade in one direction must be less numerous than dealers who weakly prefer to trade in the other direction, meeting the requirement for market clearing, since in each type of equilibrium, there is one type of dealers indifferent between holding two levels of inventory.
3 Who Buys and Who Sells?

3.1 Trading Directions

A major point of interest of our analysis is how the directions of trades among dealers having different capacities to hold the asset is determined. A direct corollary of Proposition 3 is that:

Corollary 1

(a) For \( A \leq A_S \), small dealers only sell to but do not buy from large dealers.

(b) For \( A \in (A_S, n^{LD} + \frac{n^{SD}}{2} + \frac{c}{5}) \), small dealers sell to more than they buy from large dealers if they buy from large dealers at all.

(c) For \( A \in (n^{LD} + \frac{n^{SD}}{2} + \frac{c}{5}, A_B) \), small dealers buy from more than they sell to large dealers if they sell to large dealers at all.

(d) For \( A \geq A_B \), small dealers only buy from but do not sell to large dealers.

Parts (a) and (d), respectively, are due to how small dealers only sell in the Selling Equilibrium and only buy in the Buying Equilibrium. In the Balanced-Selling Equilibrium as for Part (b), \( S_1 \)'s sell to close the gap between large dealers’ demand for and supply of the asset in the inter-dealer market. While a fraction of \( S_0 \)'s may buy given that small dealers are indifferent between holding an empty or a one-unit inventory, small dealers must sell more than they buy for market clearing. Part (c) describes the mirror opposite of Part (b). In Figure 1, for each \( e \), small dealers on balance sell to large dealers for \( A \) up to the demarcation between the Balanced-Selling and -Buying Equilibria but buy from large dealers for any larger \( A \).

Because large dealers value the first unit of inventory more than small dealers, for an asset supply at a relatively low level, the competitive inter-dealer market would first allocate the asset to large dealers as far as possible. This allocation takes place through small dealers selling to large dealers. Because small dealers value the last unit of inventory more than large dealers, for an asset supply at a relatively high level, the competitive inter-dealer market would first fill up the inventories of small dealers. This allocation takes place through small dealers buying from large dealers.
3.2 Small Dealers Provide Immediacy for Large Dealers

It seems reasonable that large dealers, to the extent that they possess a larger inventory capacity, should buy from small dealers to provide them liquidity. In our model, however, this is the case only when the asset supply is relatively scarce – just when small dealers should find it easiest to sell the asset to investors themselves.

**Definition 1** A dealer is said to provide immediacy for another dealer if the dealer sells to the other dealer when dealers meet investor-buyers more often than investor-sellers where

\[\mu(\theta) \frac{n_B}{n_B + n_S} > \mu(\theta) \frac{n_S}{n_B + n_S}\]

and buys from the other dealer when dealers meet investor-sellers more often than investor-buyers where

\[\mu(\theta) \frac{n_B}{n_B + n_S} < \mu(\theta) \frac{n_S}{n_B + n_S}\].

In our definition, a dealer provides immediacy for another dealer in the sense that it sells to the other dealer to provide inventory for him during which inventory should be in greater demand than inventory capacity and buys from the other dealer to help the dealer free up inventory capacity when inventory capacity should be more valuable than inventory. In general, the provision of immediacy, as we define it, can be thought of as helping to facilitate trades for the dealers concerned. It generalizes the notion of the provision of liquidity by also covering the situation where inventory, but not just liquidity, is in short supply.

**Proposition 4** In equilibrium, small dealers provide immediacy for large dealers.

If we can measure how quickly dealers sell in a given OTC market to their customers by, for example, measuring how long the asset stays in their inventories, we may directly test Proposition 4 with transaction-level data by checking if it is true that in a market where dealers do sell quickly, small dealers tend to sell to large dealers to help them replenish their inventories, whereas in a market where dealers may only sell slowly, small dealers tend to buy from large dealers to help them free up inventory capacities.

In the meantime, there exists indirect evidence that seems to support the Proposition as laid out in the following sections.
3.3 Pre-Crisis and Post-Crisis Trading Directions

Dealers’ overall inventory holding gradually increased in the run-up to the financial crisis in 2008, peaking at nearly $5 trillion but then declined abruptly in the aftermath of the crisis and has remained at about $3.5 trillion, the level in 2005, ever since (Adrian et al. (2017)).\footnote{See also Di Maggio, Kermani and Song (2017) and Randall (2015), among others.}

In our model, this development may be modeled as a result of a decline in dealers’ aggregate inventory capacity, which in turn may be a result of regulatory changes brought by the Dodd-Frank Act and/or dealers’ natural responses to the changing market environment. Specifically, let $n_{LD} = \gamma n^D$ and $n_{SD} = (1 - \gamma) n^D$ for some fixed $\gamma \in (0,1)$; a given change in $n^D$, as a result of a change in the market environment, is a proportionate change in dealers’ aggregate inventory capacity $n_{SD} + 2n_{LD} = (1 + \gamma) n^D$.\footnote{Aggregate dealers’ inventory capacity may also change with $\gamma$. In the Online Appendix, we repeat the analysis in this Section with respect to changes in $\gamma$ in place of $n^D$. The same results largely remain.}

**Proposition 5** A sufficient condition for dealers’ aggregate inventory holding $A^D = n^D_{SD} + n^D_{LD} + 2n^D_{LD}$ to be continuously increasing in $n^D$ is that

$$\frac{\theta \eta' (\theta)}{\eta(\theta)} < \frac{4 + 2\sqrt{2}}{4 + 3\sqrt{2}} \approx 0.83.$$ 

If an increase in $n^D$ is not followed by an increase in $A^D$, there would be more dealers having spare capacities to buy from investors. Investors sell faster as a result as long as the sufficient condition in the Proposition is met.\footnote{An increase in $n^D$, other things equal, means that there would be more dealer-buyers. This should lead to an increase in the probability that investors sell, given by $\eta(\theta) \frac{n^D_B}{n_B}$, through the increase in $n^D_B$. Any decrease in $A^D$, however, would also result in investors buying at a slower rate, giving rise to a larger $n^I_B$ and ultimately a possibly smaller $\theta = \frac{n^D_B}{n^D_B + n^I_B}$. This second order effect would never dominate for a not-too-elastic $\eta(\theta)$. By the concavity of the $\eta(\theta)$ function, $\frac{\eta'(\theta)}{\eta(\theta)}$ is restricted to be below unitary to begin with. The further restriction in the Proposition is a fairly mild sufficient condition.}

And when they do so, they hold less inventory. But then with a given asset supply, investors and dealers cannot be both holding less inventory.

A further corollary of Proposition 3 is that:

**Corollary 2** Suppose $A > \frac{1}{e \gamma^6}$. Let $n^D_{BE} < A - \frac{e}{\gamma}$ satisfy,

$$\frac{\delta}{1 - \delta} \left( n^D_{BE} + \frac{1}{\gamma^6} - A \right) (\gamma + 1) (1 - \gamma) n^D_{BE} - \mu \left( \frac{(1 - \gamma) n^D_{BE}}{A - \frac{e}{\gamma} - n^D_{BE} (1 + \gamma)} \right. = 0, \quad (27)$$

If an increase in $n^D$, other things equal, means that there would be more dealer-buyers. This should lead to an increase in the probability that investors sell, given by $\eta(\theta) \frac{n^D_B}{n_B}$, through the increase in $n^D_B$. Any decrease in $A^D$, however, would also result in investors buying at a slower rate, giving rise to a larger $n^I_B$ and ultimately a possibly smaller $\theta = \frac{n^D_B}{n^D_B + n^I_B}$. This second order effect would never dominate for a not-too-elastic $\eta(\theta)$. By the concavity of the $\eta(\theta)$ function, $\frac{\eta'(\theta)}{\eta(\theta)}$ is restricted to be below unitary to begin with. The further restriction in the Proposition is a fairly mild sufficient condition.
and $n_{SE}^D > \frac{1}{\gamma} \left( A - \frac{x}{\delta} \right)$ satisfy,

\[
\frac{\delta}{1 - \delta} \left( A - \gamma n_{SE}^D - \frac{\gamma x}{\delta} \right) (\gamma + 1) - \mu \left( \frac{(1 - \gamma) n_{SE}^D}{\frac{x}{\delta} + \gamma n_{SE}^D - A} 1 + \gamma \right) = 0.
\tag{28}
\]

(a) For $n^D \in (0, n_{BE}^D)$, the Buying Equilibrium holds.
(b) For $n^D \in \left[ n_{BE}^D, \frac{2(A - \frac{x}{\delta})}{1 + \gamma} \right)$, the Balanced-Buying Equilibrium holds.
(c) For $n^D \in \left( \frac{2(A - \frac{x}{\delta})}{1 + \gamma}, n_{SE}^D \right)$, the Balanced-Selling Equilibrium holds.
(d) For $n^D \geq n_{SE}^D$, the Selling Equilibrium holds.

For the smallest $n^D$, there are few dealers in the market who may buy from investor-sellers and hold the asset in inventory for sale to investor-buyers. In this case, all dealers in the market must hold at least a unit inventory as in the Buying Equilibrium. And then as dealers become more numerous and their overall inventory capacity goes up, the market evolves successively into equilibrium types in which dealers’ inventory capacities are used up to a lesser and lesser extent.\(^{13}\)

Proposition 5 and Corollary 2 together say that as $n^D$ falls from a presumably high level before the financial crisis to a low level afterwards, aggregate dealers’ balance sheet shrinks, while the market may evolve from an equilibrium where small dealers sell to large dealers as in the Selling or the Balanced-Selling Equilibrium to an equilibrium where small dealers buy from large dealers as in the Buying Equilibrium or the Balanced-Buying Equilibrium. All this is consistent with the finding in Adrian et al. (2017), as depicted in Figure 3 of the paper, that while large dealers expanded their balance sheets much more than small dealers did pre-crisis, a few small dealers actually expanded their balance sheets during which large dealers were downsizing theirs post-crisis. In our model, pre-crisis, while small dealers sell to large dealers, the latter expand their balance sheet to a greater extent. Post-crisis, while small dealers buy from large dealers, selected small dealers can be expanding their inventory holdings while aggregate dealers’ inventory holding shrinks.

\(^{13}\)The Corollary restricts attention to where $A > \frac{1}{\gamma} \frac{x}{\delta}$. For smaller $A$, the Buying Equilibrium never holds and then for still smaller $A$, even the Balanced Equilibrium never holds. Otherwise, the tendency that the equilibrium type changes from one where the dealers’ inventory capacities are used up to a greater to one where the capacities are used up to a lesser extent as $n^D$ increases remains. We present the full results in the Online Appendix.
3.4 Markups

With a competitive inter-dealer market, investors sell at the same price \( p_{IS} \) to and buy at the same price \( p_{IB} \) from any dealers, in addition to all inter-dealer trades taking place at just one price \( p \). This means that in a given equilibrium, all dealers earn the same markup,

\[
\rho_{DB} \equiv \frac{p - p_{IS}}{p_{IS}},
\]

from buying from investors at price \( p_{IS} \) for selling to other dealers at price \( p \) as the first links of the intermediation chain and the same markup,

\[
\rho_{DS} = \frac{p_{IB} - p}{p},
\]

as the last links from selling to investors at price \( p_{IB} \) by buying from other dealers at price \( p \). Meanwhile, there is also the same “round-trip” markup,

\[
\rho_{RT} = \frac{p_{IB} - p_{IS}}{p_{IS}},
\]

for the percentage difference between how much investor-buyers pay and how much investor-sellers receive, whoever the investors buy from and sell to. Even so, our model does have some specific implications on when these markups would be higher or lower.

3.4.1 Dealers’ markups

As the asset supply becomes more abundant, there should be fewer investor-buyers left but more investor-sellers remain in the market. This implies that:

**Proposition 6** The probability that a dealer meets an investor-buyer \( \mu(\theta) \frac{n^I_B}{n^I_B + n^I_S} \) is decreasing whereas the probability that a dealer meets an investor-seller \( \mu(\theta) \frac{n^S_L}{n^I_B + n^I_S} \) is increasing in \( A \).

The Proposition suggests that the provision of inventory by one dealer for another should be less and less valuable relative to the provision of liquidity as \( A \) increases. Then, it follows that:

**Conjecture 1** The difference between \( \rho_{DB} \) and \( \rho_{DS} \) should be decreasing in \( A \).
To evaluate the Conjecture, we calculate and then plot the two markups against $A$ in Figure 2.\footnote{We were not able to prove the Conjecture and Conjecture 2 to follow. The price equations, as detailed in the Online Appendix, are complicated equations of the the meeting probabilities, which in turn vary with $n^D_B$, $n^D_S$, $n^L_I$, and $n^S_I$ in highly non-linear manners. These variables, in turn, may only be determined implicitly. The numerical example assumes $\eta(\theta) = 1 - \exp(-\theta)$, $\epsilon = 0.15$, $\delta = 0.08$, $\beta = 0.98$, $n^{LD} = 0.5$, $n^{SD} = 0.5$ and $A$ varying from 0.1 and up. The details of the computations can be found in the Online Appendix.} When the Buying Equilibrium holds, $\rho_{DS} = \infty$ with $p = 0$, and so is left out of the plot. By changing the normalization that low-valuation investors and dealers place a positive value, instead of zero, on holding a unit of the asset, $p$ would stay positive in any equilibrium while the basic forces in our model should still contrive to give rise to a large $\rho_{DS}$ in the Buying Equilibrium. In Figure 2, indeed, the difference $\rho_{DB} - \rho_{DS}$ is decreasing in $A$, larger in the Selling and the Balanced-Selling Equilibria, smaller in the Balanced-Buying Equilibrium and the Buying Equilibrium where $\rho_{DS} = \infty$.\footnote{This result and that of Conjecture 2 to follow are robust. They hold more or less over the whole wide range of parameters that we have tried out. The details of the robustness checks are in the Online Appendix.}

Because when the Selling or the Balanced-Selling Equilibrium holds, a unit of the asset tends to stay in a dealer’s inventory for a short time while small dealers sell to large dealers on balance, we may test the joint implication of Corollary 1 and Conjecture 1 by checking if

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{Markups}
\end{figure}
during such times $\rho_{DB} - \rho_{DS}$ tends to be larger than during which a unit of the asset tends to stay in a dealer’s inventory for a long time while small dealers buy from large dealers on balance as in the Buying or the Balanced-Buying Equilibrium.

3.4.2 Round-trip Markup

In the Buying or the Balanced-Buying Equilibrium, either of which obtains with a large asset supply, there can only be a small, if not zero altogether, inter-dealer price and by extension a small price dealers are willing to pay investors for the asset $p_{IS}$. The price investor-buyers pay for the asset $p_{IB}$ would be less affected by the large asset supply though as it should be bounded from below by the expected discounted flow benefits they derive from the ownership of the asset.

**Conjecture 2** The round-trip markup $\rho_{RT}$ is higher for an $A$ that results in the market to be in either the Buying or the Balanced-Buying Equilibrium than for an $A$ that results in the market to be in either the Selling or the Balanced-Selling Equilibrium.

In the Buying Equilibrium, $p_{IS}$ is equal to zero since the inter-dealer price is equal to zero and that $p_{IS} \leq p$ must hold for dealers to willingly buy from investors. Hence, $\rho_{RT} = \infty$ in the Buying Equilibrium and the Conjecture holds trivially for that equilibrium. But just as $\rho_{DS}$ should remain large but finite in a variant of our model with a positive $p$ in the Buying Equilibrium, there should remain a large and finite $\rho_{RT}$ as well in the given model variant. The plot in Figure 2 shows that indeed $\rho_{RT}$ in the Balanced-Buying Equilibrium is above $\rho_{RT}$ in the Selling and the Balanced-Selling Equilibria.

To test Conjecture 2, as for testing Conjecture 1, we can check if $\rho_{RT}$ tends to be higher in an environment that indicates that the market is in the Buying or the Balanced-Buying Equilibrium.

In the meantime, there exists indirect evidence that sheds light on the validity of the two conjectures as we shall discuss next.
3.5 Core-Periphery Trading Structure

In many OTC markets, dealers can be distinguished by their degrees of centrality where more central dealers are those who trade with a greater number of other dealers. This distinction resembles a core-periphery structure in the abstract, where the more central dealers can be thought of as dealers in the core who may trade with just any other dealers and the less central dealers can be thought of as dealers in the periphery who only trade with the core dealers (Li and Schürhoff (2019) and Hollifield et al. (2017)).

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Table 2: Dealers Chains

In our model, the possible dealer chains, with $C$ denoting a customer, $SD$ a small dealer and $LD$ a large dealer, by the entries in Table 1, are as depicted in Table 2. The Table shows that, as we argue before, small dealers only trade with large dealers, but not among themselves, whereas large dealers trade among themselves as well as with small dealers. Over a given time interval then, a given large dealer should trade with more dealers than a given small dealer. The model large dealers thus behave similarly as the more central dealers in the real world OTC market do. The model small dealers, to the extent that they only trade with large dealers, behave similarly as the less central dealers in reality do. Under this interpretation, our model says that the less central dealers trade to provide immediacy for the more central ones. This implication is counterintuitive but there exists empirical evidence consistent with the implications of Conjectures 1 and 2, in addition to the finding in Li and Schürhoff (2019) that central dealers tend to provide liquidity in times of market stress.

---

16 Here, we omit any trades merely involving two dealers switching identities. Those trades are spurious in that they do not benefit either dealer and we think it is not far-fetched to presume that they should not take place and indeed they would not take place with a small trading cost in place or if there is a small information asymmetry as shown in Bethune, Sultanum and Trachter (2016).

17 Li and Schürhoff (2019) find that central dealers tend to hold more assets in inventory. In all three types of equilibrium in our model, the optimal inventory level for a large dealer is at least weakly higher than that of a small dealer.
3.5.1 Dealers’ Markups

Hollifield et al. (2015) report in their Table 6 that the percentage bid-ask spreads peripheral dealers earn when they are the last links of the intermediation chain, buying from another dealer for selling to an investor, are higher than when they are the first links, buying from an investor for selling to other dealers. Meanwhile, there are no statistically significant differences between the two spreads for central dealers.

By Table 2, in our model, the small peripheral dealers are the last links of the intermediation chain in the Buying and the Balanced-Buying Equilibria, earning the markup $\rho_{DS}$ but are the first links in the Selling and the Balanced-Selling Equilibria, earning the markup $\rho_{DB}$. By Conjecture 1, $\rho_{DS}$ is largest relative to $\rho_{DB}$ in the former set of equilibrium. Moreover, as can be seen in Figure 2, $\rho_{DS}$ does exceed $\rho_{DB}$ then and only then (the solid segment of the $\rho_{DS}$ curve versus the solid segment of the $\rho_{DB}$ curve), consistent with the finding in Hollifield et al. (2015).

Intuitively, when the market is in either the Buying or the Balanced-Buying Equilibrium, as arising from a large asset supply, dealers would be selling to investors in a slow market in which it takes a long time on average for a dealer to sell a unit. The intermediation service a small dealer provides for a large dealer by selling to investors on his behalf should then command a high return. When the market is in either the Selling or the Balanced-Selling Equilibrium, as arising from a small asset supply, dealers would be buying from investors in a market populated by a large number of dealer-buyers versus a small number of investor-sellers. In this market, there should only be a small return earned by a small dealer from intermediating the sale by an investor since the dealer has a relatively weak bargaining position vis-à-vis an investor having plenty of other meeting opportunities.

The large central dealers in our model, as shown in Table 2, can be the first links, earning the markup $\rho_{DB}$, or the last links, earning the markup $\rho_{DS}$, of a dealer chain in any equilibrium type. In Figure 2, they earn a higher markup as the first links than the last links in the earlier part of the Selling Equilibrium but a lower markup as the first links than the last links in other equilibrium types. If the sample in Hollifield et al. (2015) is composed of markets in all three equilibrium types, there can certainly be no statistically significant differences between
the two observed markups large dealers earn, just as what the paper reports.

3.5.2 Round-Trip Markup

Li and Schürhoff (2019) find that the round-trip markup $\rho_{RT}$ tends to be higher when the first link of the dealer chain is a central dealer. In our model, by Conjecture 2, $\rho_{RT}$ is higher in the Buying and the Balanced-Buying Equilibria. In the Buying Equilibrium in which all small dealers hold a full inventory, the first link of a dealer chain must be a large dealer. In the Balanced-Buying Equilibrium, any small dealers who buy from an investor would skip the inter-dealer market to sell directly to an investor afterwards.

3.5.3 Liquidity Provision for Investors by Central Dealers

Li and Schürhoff (2019) argue in their Hypothesis 4 and then verify that central dealers are liquidity providers of last resort in that investors trade with central dealers more during times of market stress, which, in our model, correspond to the market in the Balanced-Buying or the Buying Equilibrium, either of which obtains with a large asset supply relative to demand and/or with a small overall dealers’ inventory capacity. In the Balanced-Buying Equilibrium, less than one-half of all small dealers still have any capacity to buy from investors. Any investors who sell are likely to sell to a large central dealer. In the Buying Equilibrium, no small peripheral dealers have any capacity to buy at all. Any investors who sell may only sell to a large central dealer.

That central dealers are the liquidity providers of last resort is not inconsistent with less central dealers providing immediacy for central dealers – the less central dealers help the more central dealers free up inventory capacities to provide liquidity for investors during times of market stress.

3.5.4 Persistence of Trading Directions

Li and Schürhoff (2019) find that given that there is a directional (buy or sell) trade between two dealers in one month, the probability that the same directional trade remains in the next month is 62%. Our analysis has a precise and readily testable implication as to when the trading
direction is persistent. If the small dealers in our model are less central and large dealers more central dealers, in our model, the trading direction between a more central and a less central dealers is persistent while the direction between two central dealers is not persistent.

4 Other Implications

4.1 Inventory Expansion

A further implication of Proposition 3 is that:

**Corollary 3** *In the new steady state after a positive shock to the asset supply, only large dealers expand inventory in the Selling Equilibrium, only small dealers expand inventory in the Balanced Equilibrium, and again only large dealers expand inventory in the Buying Equilibrium.*

In general, Corollary 3 says that, except for a sufficiently large shock to the asset supply, an entire group of dealers would not adjust their inventory holdings in response. Rather counterintuitively, individual dealers’ responses to the changing market environment can become sluggish, despite or rather because of competition in the inter-dealer market. One possibility to directly test the implication is to check whether or not and to what extent that there is an entire category of dealers not expanding their inventory holdings amidst a not-too-large new bond issue.

4.2 Inter-dealer Trading Volume

Strictly speaking, for a given type of equilibrium, the inter-dealer market trading volume is indeterminate given the existence of a continuum of equilibrium. Any but one of the continuum, however, involves trades between two dealers merely switching identities. In studying the model’s implications on trading volumes, there is good reason to focus solely on the equilibrium with the least trades, in which any trades that take place are trades carried out by inframarginal buyers or sellers to rebalance inventories. The trading volume \( TV \) in the

\[ 18 \text{ As laid out in note 16.} \]
Selling, Balanced, and the Buying Equilibria are then given by, respectively,

\[
TV = m_1^{SD} + m_2^{LD},
\]

\[
TV = \begin{cases} 
  m_0^{LD} & A \leq n^{LD} + \frac{n^{SD}}{2} + \frac{\epsilon}{2} \\
  m_2^{LD} & A \geq n^{LD} + \frac{n^{SD}}{2} + \frac{\epsilon}{2} 
\end{cases},
\]

\[
TV = m_0^{SD} + m_0^{LD}.
\]

**Proposition 7** The inter-dealer market trading volume changes non-monotonically with \( A \):

(a) first increasing in the Selling Equilibrium,

(b) becomes decreasing in the Balanced-Selling Equilibrium,

(c) then turns increasing again in the Balanced-Buying Equilibrium,

(d) and eventually becomes decreasing in the Buying Equilibrium.

That is, the trading volume peaks at where the Selling Equilibrium turns into the Balanced Equilibrium and when the Balanced Equilibrium turns into the Buying Equilibrium.

The Proposition says the trading volume is M-shaped and that the inter-dealer market is most active when the asset supply is at a relatively low level but not at the lowest level and at a relatively high level but not at the highest level. The intuition lies in how the measures of buyers and sellers vary with the asset supply.\(^{19}\) Inter-dealer trading should be most active when both buyers and sellers abound. For the smallest asset supply, there can only be few dealers possessing any inventory to sell in the inter-dealer market; for the largest asset supply, there can only be few dealers having any spare capacity to buy in the market. For intermediate levels of \( A \), dealers’ need to rebalance inventories is minimal as they buy from and sell to investors with more or less the same ease. Thus, there cannot be much inter-dealer trading when the asset supply is either lowest, highest, or right in between.

5 Efficient Decentralized Market Trades

Might the provision of immediacy by small for large dealers in equilibrium be efficient or otherwise? In this Section, we study the efficient allocation of inventory among dealers and

\(^{19}\)The same intuition is known in the literature to explain bell-shaped transaction volumes. See, for example, Kiyotaki and Wright (1989) and Hugonnier et al. (2018).
compare it with the equilibrium allocation.

A social planner maximizes investors’ flow payoffs from the ownership of the asset given by,

\[ W = \max \left\{ \sum_{t=0}^{\infty} \beta^t n_H^Q(t) \nu \right\}, \tag{29} \]

subject to the same search and matching frictions that agents face.

A priori, the equilibrium trades in the frictional investor-dealer market are constrained efficient where any trades with a positive surplus, but only such trades, will take place. Specifically, any trade between an investor-buyer and a dealer-seller is efficient with the former, but not the latter, deriving the flow payoff \( \nu \) from holding a unit of the asset. But then a dealer-seller becomes a dealer-seller in the first place only by acquiring the asset from an investor-seller. Then, any and all trades between an investor-seller and a dealer-buyer are also efficient. This means that it suffices for us to ask how the planner may wish to allocate the inventory of asset among the dealers after each round of investor-dealer trades and whether the allocation coincides with the allocation that falls out from the inter-dealer market in equilibrium.

**Proposition 8** In the steady state, the allocation from the decentralized market trades coincides with the planning optimum.

In the planning optimum, inventories are allocated to dealers to enable high-valuation investors to acquire the asset most rapidly and to enable units of the asset to be transferred from low-valuation investors to dealers the quickest. In the competitive inter-dealer market, inventories and spare capacities are allocated to dealers who value them the most – the very dealers who have the best use of them for trading with investors. Perhaps not surprisingly, the equilibrium allocation coincides with the constrained optimum allocation given that dealers trade an indivisible asset in a competitive inter-dealer market. More interestingly, this means that for efficiency, the small peripheral dealers should trade to provide immediacy for the large central dealers.\(^{20}\)

\(^{20}\) If investors’ portfolio decisions are on the intensive margin where they choose how many units of the asset to hold, see Lagos and Rocheteau (2006) for an example, trades in the investor-dealer markets are inefficient due to a holdup problem introduced by bargaining. In such a model, it would be natural to allow dealers to
6 Extension and Robustness

Our major results – Proposition 1 and the implications thereof – seemingly rest on a number of simplifying assumptions that may appear ad hoc. In the following, we discuss how the Proposition and its implications survive two generalizations that seem most warranted. In Appendix 8.3, we discuss an additional extension that further adds to the generality of the analysis.

6.1 Matching Opportunity

If each large dealer can hold up to two units in inventory and may possess up to two units of spare inventory capacity, perhaps an equally plausible assumption is that they can meet up to two investors in each period. We shall demonstrate below how the ranking of the marginal benefits of inventory in Proposition 1 can be left intact.

First, if a large dealer has up to two matching opportunities with investors, a reasonable matching technology should be such that

A1. the probability that a large dealer meets at least one investor-seller (-buyer) is weakly higher than the probability that a small dealer meets one investor-seller (-buyer).

If, in addition, the matching technology exhibits diminishing returns in the sense that

A2. the probability that a large dealer meets two investor-sellers (-buyers) is weakly lower than the probability that a small dealer meets one investor-seller (-buyer), then the ranking in Proposition 1 remains.

First, consider the costs and benefits of acquiring the first unit of inventory in the inter-dealer market for the two types of dealers. Filling up the first unit of capacity is costly to a small dealer as long as he shall meet one investor-seller in the next period but is costly to a large dealer only if he meets two investor-sellers in the next period since a large dealer meeting

also choose interior inventory levels by assuming, for example, a quadratic inventory holding cost as in Sannikov and Skrzypacz (2016) in which dealers differ in their degree of inventory aversion. Dealers with large inventory aversion would endogenously have a small inventory capacity. As dealers Nash bargain with investors, the equilibrium inventory levels and the efficient inventory levels could be different. In this case, both the size and direction of the inter-dealer trades can be different in equilibrium and in the efficient solution. A detailed analysis is beyond the scope of this paper.
only one investor-seller still has capacity to buy from the investor. By A2, the expected cost is higher for the small dealer. The expected benefit is higher for the large dealer – if A1 holds, the large dealer can sell the unit at a weakly higher probability. Then, $V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD}$ should follow. The inequality should be strict if either one of the relation in A1 or A2 is strict.

Next, consider the costs and benefits of using up the last unit of spare capacity for the two types of dealers. Exhausting one’s capacity is costly to a dealer as long as the dealer shall meet one or more investor-seller in the next period. By A1, the expected cost is higher for the large dealer. A small dealer benefits from the additional unit of inventory if he meets one investor-buyer while a large dealer benefits only if he meets as many as two investor-buyers. If A2 holds, the expected benefit is higher for the small dealer. Then, $V_{1}^{SD} - V_{0}^{SD} \geq V_{1}^{LD} - V_{0}^{LD}$ should follow. The inequality should be strict if either one of the relation in A1 or A2 is strict.

As an example of how Assumptions A1 and A2 above can be satisfied, suppose that each large dealer participates as two independent agents in the matching process and thereby earns two independent matching opportunities. In this case, market tightness is given by

$$\theta = \frac{n_{SD} + 2n_{LD}}{n_{S} + n_{B}}.$$ 

Notice that

$$1 - \left(1 - \mu(\theta) \frac{n_{S}}{n_{B} + n_{S}}\right)^{2} > \mu(\theta) \frac{n_{S}}{n_{B} + n_{S}} > \left(\mu(\theta) \frac{n_{S}}{n_{B} + n_{S}}\right)^{2},$$

where the first term is the probability that a large dealer meets at least one investor-seller, the second term the probability that a small dealer meets one investor-seller and the last term the probability that a large dealer meets two investor-sellers. Both Assumptions A1 and A2 hold true. In general, the two inequalities above would only fail to hold if the two matching outcomes for the large dealer are perfectly and positively correlated events – the large dealer either meets two investor-sellers or not a single one at the same probability that a small dealer meets one investor-seller or not, in which case all three probabilities in (30) become one and the same.\textsuperscript{21}

\textsuperscript{21}It is perhaps of interest to note that in a model where large dealers have two meeting opportunities, the inter-dealer price in the Buying Equilibrium should stay positive even if dealers and low-valuation investors do not value the asset.
6.2 Frictional Inter-dealer Market

There exists ample evidence that the inter-dealer market is better described as a decentralized market (Li and Schürhoff (2019) and Henderschott, Li, Livdan and Schürhoff (2017)), where it takes time and effort for dealers to find trading partners, in which case dealers, by all means, have incentives to manage inventory for future trading needs as dealers in our model, who only have periodic access to the competitive inter-dealer market, do. Where the incentives are similar, the major implications of our model should survive in a model of a frictional inter-dealer market.

In the following, we report the results of our analysis of a model of a frictional inter-dealer market but is otherwise identical to the main model of the paper. The model is set in continuous time as it is a more convenient setting for a model in which both the investor-dealer and the inter-dealer markets are decentralized. With a frictional inter-dealer market, in addition to meeting an investor-buyer at the rate \( \mu(\theta) \frac{n_B}{n_B+n_S^I} \) and an investor-seller at the rate \( \mu(\theta) \frac{n_S^I}{n_B+n_S^I} \), a dealer meets another randomly selected dealer at a fixed rate \( \alpha \) per unit of time. The terms of trade between two dealers are determined by Nash Bargaining, as are prices in the investor-dealer market. The value functions and equilibrium conditions are presented in Appendix 8.2. All notations for the revised model have the same meanings as for the main model.

**Proposition 9** \( V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD} \) in any steady-state equilibrium involving trades between investors and dealers. The two equalities are strict if and only if the surplus for the I_B-L_1 match \( z_{I_B,L_1} \equiv U_H^{ON} - U_B^{I} - (V_1^{LD} - V_0^{LD}) > 0 \).

There is thus the same ranking of the marginal benefits of inventory as in Proposition 1, which again implies that profitable bilateral trades between two dealers only include those between an \( L_0 \) and an \( S_1 \) (by the first inequality in the Proposition), between an \( S_0 \) and an \( L_2 \) (by the second inequality), and between an \( L_0 \) and an \( L_2 \) (by the two inequalities together).

In the competitive inter-dealer market of the main model, small dealers on balance sell to (buy from) large dealers when the asset supply is meagre (abundant). Below, we set out to verify that the same tendency remains in the frictional inter-dealer market, despite small
dealers selling to and buying from large dealers at all levels of asset supply in the market, where there are positive surpluses from all such trades.

In our numerical analyses,\(^\text{22}\) we find that the volume of small dealers’ sales to large dealers and the volume of small dealers’ purchases from large dealers, given by, respectively,\(^\text{23}\)

\[
SD_s = \alpha \frac{n_1^{SD} n_0^{LD}}{n_D},
SD_b = \alpha \frac{n_0^{SD} n_2^{LD}}{n_D},
\]

are both bell-shaped functions of \(A\) as shown in Figure 3(a).\(^\text{24}\)

\(^{22}\)It does not seem possible to analytically solve the model and to derive conditions on model fundamentals for the existence and uniqueness of equilibrium from the equilibrium conditions as detailed in Appendix 8.2. Assuming a Walrasian inter-dealer market simplifies considerably through the restrictions on \(n_i^{SD}\) and \(n_i^{LD}\) in Table 1 and thereby enables us to derive a rich set of analytical results. The numerical example uses the same parameterization used in the plots in Figures 2, except for the \(\eta(\theta)\) function, which is now assumed to be \(\eta(\theta) = \theta^{0.5}\). The change is necessitated by \(\eta(\theta)\) being a meeting rate function instead of a meeting probability function as in the main model. The details of the computations are in the Online Appendix.

\(^{23}\)The equation for \(SD_s\) obtains as an \(S_i\) meets another dealer at the rate \(\alpha\) and a fraction \(n_0^{LD}/n_D\) of those dealers are \(L_0s\). The equation for \(SD_b\) is constructed similarly.

\(^{24}\)The intuition for why the curves are bell-shaped is the same as in Hugonnier, Lester and Weill (2018) and
The Figure also shows that $SD_s > SD_b$ for smaller $A$s and vice versa.\textsuperscript{25} Hence, even though small dealers do buy from and sell to large dealers for all levels of $A$, they sell to more than they buy from large dealers when the asset supply is at a low level but buy from more than they sell to the latter when the asset supply is at a high level, just as in the competitive inter-dealer market model. This happens because small dealers only trade with large dealers in the frictional inter-dealer market also. In particular, given that $S_1$s only sell to $L_0$s whereas $L_2$s sell to $S_0$s in addition to $L_0$s, the ratio $n_1^{SD}/n_2^{LD}$ is largest when $S_0$s are most plentiful since in this case there would be fewest dealers remaining as $L_2$s relative to $S_1$s. In turn, $S_0$s are most plentiful amidst a small asset supply. And so in this environment, $SD_s$ tends to exceed $SD_b$. On the other hand, given that $S_0$s only buy from $L_2$s, whereas $L_0$s buy from $S_1$s in addition to $L_2$s, the ratio $n_0^{LD}/n_0^{SD}$ is smallest when $S_1$s are most plentiful since in this case there would be fewest dealers remaining as $L_0$s relative to $S_0$s. In turn, $S_1$s are most plentiful amidst a large asset supply for the additional units of the asset to be in circulation. And so in this environment, $SD_s$ tends to fall below $SD_b$.

In view of Figure 3(a), in which both $SD_s$ and $SD_b$ peak near each other around some intermediate level of asset supply, the aggregate trading volume, equal to the sum of $SD_s$, $SD_b$, and the volume of $L_0$-$L_2$ trades $\left(\alpha n_0^{LD} n_0^{LD} n_2^{LD} n_2^{LD}/n_1^{SD} n_1^{SD} n_2^{SD} n_2^{SD}\right)$, should likewise be single-peaked, as depicted in Figure 3(b). In our main model, the aggregate trading volume, however, is M-shaped.

This distinction between the two models suggests an empirical test for the efficiency of a given inter-dealer market. If the competitive inter-dealer market is efficient – as we have shown in Proposition 8 – aggregate trading volume should not reach the maximum level at some intermediate level of asset supply. In the Balanced Equilibrium of the competitive inter-dealer market, large dealers should only trade with small dealers to the extent that the market cannot clear otherwise. In the frictional inter-dealer market, not all possible trades between an $L_0$ and an $L_2$, which yield the highest surplus among all inter-dealer trades, can take place and a unit of the asset may only be passed on from an $L_2$ to an $L_0$ over time through a small dealer. Such roundabout trades are most numerous at an intermediate asset supply where both

\textsuperscript{25}These results are from assuming $n^{SD} = n^{LD}$. As checks for robustness, we find the same qualitative results hold with $n^{SD} > n^{LD}$ and $n^{SD} < n^{LD}$. 

Kiyotaki and Wright (1989).
$S_0$s and $S_1$s are relatively abound. The abundance of inter-dealer trades in a given market at some intermediate asset supply can thus be a sign of search frictions giving rise to a myriad of roundabout trades that would not have taken place otherwise.

A further test of the efficiency of the inter-dealer market follows from Corollary 3 and Proposition 8. In a competitive inter-dealer market, at any one time, either large or small dealers, but not both, would expand inventory in response to a given small change in the market environment. In a frictional inter-dealer market, both large and small dealers, as a group, would expand inventory holdings, in response to the same change in the market environment because not all mutually beneficial $S_0-L_2$ trades and $L_0-S_1$ trades may take place due to the search frictions in the market. The degree at which such changes are observed to take place in a given market is indicative of the inefficiency of the market.

Assuming competition in the inter-dealer market simplifies the analysis considerably to allow us to derive a host of arguably interesting results. The model, however, cannot be used to address questions of what determines the differential markups different dealers earn from trading with their investor customers in a given equilibrium. In a frictional inter-dealer market, the terms of trade between an investor and a dealer would differ among dealers of different inventory capacities and actual inventory holdings, in addition to inter-dealer prices depending on who buys and who sells.

7 Concluding Remarks

In our model, small dealers sell to large dealers when large dealers need inventory the most and buy from large dealers when large dealers need spare capacity the most. If the large dealers are interpreted as the central dealers and the small dealers are interpreted as the peripheral dealers in a core-periphery trading structure, our analysis suggests that the peripheral dealers trade to provide immediacy for the central dealers, contrary to the common conception of the roles the two types of dealers should play in inter-dealer trading.
8 Appendix

8.1 The equations for $m_{1SD}$ and $m_{1LD}$

\[ m_{1SD} = \mu(\theta) \frac{n^I_S}{n^I_S + n^I_B} n^SD_1 + \left(1 - \mu(\theta) \frac{n^I_B}{n^I_S + n^I_B}\right) n^SD_1, \quad (31) \]

\[ m_{0LD} = \left(1 - \mu(\theta) \frac{n^I_S}{n^I_S + n^I_B}\right) n^LD_0 + \mu(\theta) \frac{n^I_B}{n^I_S + n^I_B} n^LD_1, \quad (32) \]

\[ m_{1LD} = \mu(\theta) \frac{n^I_S}{n^I_S + n^I_B} n^LD_0 + \left(1 - \mu(\theta) \frac{n^I_S}{n^I_S + n^I_B}\right) \left(1 - \mu(\theta) \frac{n^I_B}{n^I_S + n^I_B}\right) n^LD_1 + \mu(\theta) \frac{n^I_B}{n^I_S + n^I_B} n^LD_2, \quad (33) \]

\[ m_{2LD} = \mu(\theta) \frac{n^I_S}{n^I_S + n^I_B} n^LD_0 + \left(1 - \mu(\theta) \frac{n^I_S}{n^I_S + n^I_B}\right) \left(1 - \mu(\theta) \frac{n^I_B}{n^I_S + n^I_B}\right) n^LD_1 + \mu(\theta) \frac{n^I_B}{n^I_S + n^I_B} n^LD_2. \quad (34) \]

8.2 The Frictional Inter-Dealer Market Model

The model is set in continuous time in which all agents discount the future at the rate $r$ and a dealer meets another randomly selected dealers at the rate $\alpha$ per time unit. All other notations have the same meanings as for the main model.

8.2.1 Value Functions

To define the value functions, we rule out a priori any exchanges between two dealers that merely result in the two dealers concerned switching states as such exchanges cannot give rise to a positive surplus.

Small dealers An $S_0$, who can only buy, meets an investor-seller at the rate $\eta(\theta) \frac{n^I_S}{n^D}$ and another dealer at the rate $\alpha$. Among all dealers that the $S_0$ may meet, there can be a potentially profitable exchange only if the counterparty is an $L_1$ or an $L_2$. If all investor-dealer trades yield non-negative surpluses, as we verify in the proof of Proposition 9,

\[
rV_0^{SD} = \eta(\theta) \frac{n^I_S}{n^D} (V_1^{SD} - V_0^{SD} - p_{S_0,L_2}) + \alpha \left\{ \frac{n^I_B}{n^D} \max \{-p_{S_0,L_1} + V_1^{SD} - V_0^{SD}, 0\} \right. \\
+ \left. \frac{n^I_B}{n^D} \max \{-p_{S_0,L_2} + V_1^{SD} - V_0^{SD}, 0\} \right\},
\]
where \( p_{b,s} \) denotes the terms of exchange between buyer \( b \) and seller \( s \). An \( S_1 \), who can only sell, meets an investor-buyer at the rate \( \eta(\theta) \frac{n_1^I}{n_B} \). The \( S_1 \) may also sell to an \( L_0 \) or an \( L_1 \). Then,

\[
r V_1^{SD} = \eta(\theta) \frac{n_B^I}{n_D} (p_{I_B,S_1} + V_0^{SD} - V_1^{SD}) + \alpha \left\{ \frac{n_0^{LD}}{n_D} \max \{p_{L_0,S_1} + V_0^{SD} - V_1^{SD}, 0\} \\
+ \frac{n_1^{LD}}{n_D} \max \{p_{L_1,S_1} + V_1^{SD} - V_0^{SD}, 0\} \right\}.
\]

**Large dealers** An \( L_0 \) may buy from an investor-seller, an \( S_1 \), or an \( L_2 \). Then,

\[
r V_0^{LD} = \eta(\theta) \frac{n_S^I}{n_D} (V_1^{LD} - V_0^{LD} - p_{L_0,I_S}) + \alpha \left\{ \frac{n_1^{SD}}{n_D} \max \{-p_{L_0,S_1} + V_1^{LD} - V_0^{LD}, 0\} \\
+ \frac{n_2^{LD}}{n_D} \max \{-p_{L_0,L_2} + V_1^{LD} - V_0^{LD}, 0\} \right\}.
\]

An \( L_1 \) may buy from an investor-seller and sell to an investor-buyer. Among dealers, he may sell to an \( S_0 \), buy from an \( S_1 \), and either buy from or sell to another \( L_1 \). Then,

\[
r V_1^{LD} = \eta(\theta) \frac{n_L^I}{n_S} (V_1^{LD} - V_0^{LD} - p_{L_1,I_S}) + \eta(\theta) \frac{n_I^I}{n_D} (p_{I_B,L_1} + V_0^{LD} - V_1^{LD}) \\
+ \alpha \left\{ \frac{n_0^{SD}}{n_D} \max \{p_{S_0,L_1} + V_0^{LD} - V_1^{LD}, 0\} + \frac{n_1^{SD}}{n_D} \max \{-p_{L_1,S_1} + V_2^{LD} - V_1^{LD}, 0\} \\
+ \frac{n_2^{LD}}{n_D} \max \{-p_{L_1,L_1} + V_2^{LD} - V_1^{LD}, 0\} \right\}.
\]

An \( L_2 \), who can only sell, meets an investor-buyer at the rate \( \eta(\theta) \frac{n_L^I}{n_B} \). Among dealers, he may sell to an \( S_0 \) or an \( L_0 \). Then,

\[
r V_2^{LD} = \eta(\theta) \frac{n_I^I}{n_B} (p_{I_B,L_2} + V_1^{LD} - V_2^{LD}) + \alpha \left\{ \frac{n_0^{SD}}{n_D} \max \{p_{S_0,L_2} + V_1^{LD} - V_2^{LD}, 0\} \\
+ \frac{n_1^{LD}}{n_D} \max \{p_{L_0,L_2} + V_1^{LD} - V_2^{LD}, 0\} \right\}.
\]

**Investors** An investor-buyer may buy from an \( S_1 \), an \( L_1 \), or an \( L_2 \). Then,

\[
r U_B^I = \eta(\theta) \frac{n_B^D}{n_B} \left( U_H^{ON} - U_B^I - \frac{n_1^{SD}}{n_S} p_{I_B,S_1} - \frac{n_1^{LD}}{n_B} p_{I_B,L_1} - \frac{n_2^{LD}}{n_B} p_{I_B,L_2} \right) - \delta U_B^I,
\]

where

\[
r U_H^{ON} = v + \delta \left( U_S^I - U_H^{ON} \right).
\]

An investor-seller may sell to an \( S_0 \), an \( L_0 \), or an \( L_1 \). Then,

\[
r U_S^I = \eta(\theta) \frac{n_B^D}{n_B} \left( \frac{n_1^{SD}}{n_B} p_{S_0,I_S} + \frac{n_0^{LD}}{n_B} p_{L_0,I_S} + \frac{n_1^{LD}}{n_B} p_{L_1,I_S} - U_S^I \right).
\]
8.2.2 Equilibrium Conditions

To begin, since, by Proposition 9, among all dealers, $S_1$s only sell to $L_0$s whereas $S_0$s only buy from $L_2$s, in the steady state in which $n_0^{SD} = n_1^{SD} = 0$,

$$n_1^{SD} (\eta(\theta)n_B^I + \alpha n_0^{LD}) = n_0^{SD} (\eta(\theta)n_S^I + \alpha n_2^{LD}). \tag{35}$$

Also, where $L_1$s do not trade in the inter-dealer market and that $L_0$s buy from $S_1$s and $L_2$s, the equations for $n_0^{LD} = 0$ becomes,

$$n_1^{LD} \eta(\theta)n_B^I = (n^{LD} - n_1^{LD} - n_2^{LD}) (\eta(\theta)n_S^I + \alpha n_1^{SD} + \alpha n_2^{LD}). \tag{36}$$

The continuous time versions of the respective steady-state conditions for $n_H^O = 0, n_S^I = 0$, and $n_B^I = 0$ are given by

$$n_B^I \eta(\theta) \frac{n_S^D}{n_B^I} = \delta n_H^O, \tag{37}$$

$$\delta n_H^O = \eta(\theta) \frac{n_S^D}{n_B^I} n_S^I, \tag{38}$$

$$e = \left( \delta + \eta(\theta) \frac{n_S^D}{n_B^I} \right) n_B^I, \tag{39}$$

whereas the adding up constraint (22) remains valid.

To begin, we first use (37) and (39) to solve for

$$n_H^O = \frac{e}{\delta} - n_B^I, \tag{40}$$

while rewriting (39) as

$$n_S^D = \frac{e - \delta n_B^I}{\eta(\theta) n_B^I} n_B^I. \tag{41}$$

Then, by (37) and (38),

$$n_B^D = \frac{e - \delta n_B^I}{\eta(\theta) n_S^I} n_B^I. \tag{42}$$

Substituting (40) and (41) into (22) and by virtue of $n_S^D = n_1^{SD} + n_1^{LD} + n_2^{LD}$,

$$n_2^{LD} = A - \frac{e}{\delta} - n_S^I + n_B^I - \frac{e - \delta n_B^I}{\eta(\theta) n_B^I} n_B^D. \tag{43}$$

Because $n_B^D = n_0^{SD} + n_0^{LD} + n_1^{LD}$,

$$n_1^{SD} = n_D - n_2^{LD} - n_B^D$$

$$= n_D - A + \frac{e}{\delta} + n_S^I - n_B^I + \frac{e - \delta n_B^I}{\eta(\theta) n_B^I} n_B^D \left( \frac{1}{n_B^I} - \frac{1}{n_S^I} \right), \tag{44}$$
where the second line is from substituting from (42) and (43). Where \( n_0^{SD} = n^{SD} - n_1^{SD} \),

\[
n_0^{SD} = -n^{LD} + A - \frac{e}{\delta} - n^n_S + n^n_B - \frac{e - \delta n^n_B n^D}{\eta(\theta)} n^D \left( \frac{1}{n_B^n} - \frac{1}{n_S^n} \right).
\]

Finally, substituting (40), (43), and (44) into (22),

\[
n_1^{LD} = -n^D + \frac{e - \delta n^n_B n^D}{\eta(\theta)} n^D \left( \frac{1}{n_S^n} + \frac{1}{n_B^n} \right).
\]

To complete the characterization, we bring in the definition of market tightness from (1). Then, (35) and (36) are in terms of \( n_B^I \) and \( n_S^I \) only.

8.3 Larger Inventory Capacity for Small Dealers

Perhaps it seems trivial that, in our model, small dealers, in having just one unit of inventory capacity, never gain from trading among themselves. The question then is if and how the trading directions in Corollary 1 survive the generalization where small dealers each possess more than a unit of inventory capacity and thereby may gain by trading with one another.

Consider, in particular, that the small dealers each possess a two-unit inventory capacity, while the large dealers each possess a three-unit inventory capacity. These larger capacities are relevant only if a dealer may buy and sell up to two units of the asset in a period. The simplest extension is to assume that there are two types of investors – small investors who may each hold zero or one unit and large investors who may each hold zero or two units, and that dealers meet investors randomly independent of dealers’ types. In this environment, a dealer-seller (-buyer) holding a one-unit inventory (spare capacity) can only sell to (buy from) small investors, whereas a dealer-seller (-buyer) holding at least a two-unit inventory (spare capacity) can sell to (buy from) large investors as well as small investors.

A ranking of the marginal benefits of inventory similar to that in Proposition 1 should remain – an additional unit of the asset should be valued higher by a large dealer than by a small dealer at the same initial level of inventory for the two dealers since the former would retain a greater spare capacity for future buying needs than the latter in using up a unit of capacity. On the other hand, the large dealer should value an additional unit of the asset less than the small dealer when they start with the same spare capacity, as the former has a larger
initial inventory than the latter beforehand. The ranking of the marginal benefits of inventory in Proposition 1 may then be generalized to

\[ V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD} \geq V_2^{SD} - V_1^{SD} \geq V_3^{LD} - V_2^{LD}, \]

where the inter-dealer market clears in general only at \( p \) equal to \( \beta \) times one of the above marginal values. We next proceed to inquire how the trading directions between small and large dealers remain persistent while small dealers trade to provide immediacy for large dealers. For the ease of exposition and brevity and without loss of generality, we assume, for the following, that dealers who do not gain from trade and whose trades are not required for market clearing do not trade in the inter-dealer market.

Case 1a \( p = \beta (V_1^{LD} - V_0^{LD}) \). The buyers in the inter-dealer market are comprised of a fraction of \( L_0s \) and the sellers are \( S_1s, S_2s, L_2s, \) and \( L_3s \).

Case 1b \( p = \beta (V_2^{SD} - V_0^{SD}) \) with a fraction of \( S_1s \) selling in the inter-dealer market. The buyers in the inter-dealer market are all of \( L_0s \) and the sellers are a fraction of \( S_1s, S_2s, L_2s, \) and \( L_3s \).

For \( p \) to settle at the highest or the second highest marginal values of inventory, the two cases above should hold for the smallest \( A \). Given that all small dealers who trade in the inter-dealer market (\( S_1s \) and \( S_2s \)) sell and they sell to \( L_0s \), small dealers trade to provide immediacy for large dealers and the trading direction between small and large dealers is persistent.

Case 2 \( p = \beta (V_2^{SD} - V_0^{SD}) \) with a fraction of \( S_0s \) buying in the inter-dealer market. The buyers in the inter-dealer market are all of \( L_0s \) and a fraction of \( S_0s, S_2s, L_2s, \) and \( L_3s \). When this type of equilibrium first starts to hold, the fraction of \( S_0s \) who buy is arbitrarily close to zero. When this equilibrium turns into the equilibrium at \( p = \beta (V_2^{LD} - V_1^{LD}) \) so that all \( S_0s \) are buying, as we shall demonstrate below in the next case, there would be as many \( S_0s \) as \( S_2s \). In between, we conjecture that there remains fewer \( S_0 \) buyers than \( S_2 \) sellers. Then, on balance, small dealers are selling to and thus are still providing immediacy for large dealers at a \( p \) that should hold for a relatively small \( A \). The trading direction, though not perfectly, is largely persistent.
**Case 3** \( p = \beta (V_2^{LD} - V_1^{LD}) \). The buyers in the inter-dealer market are all of \( L_0s, S_0s, \) and possibly a fraction of \( L_1s. \) The sellers are \( S_2s, L_3s, \) and possibly a fraction of \( L_2s. \) Given that all small dealers leave the inter-dealer market with one unit of inventory whereas large dealers do so with either one or two units of inventory, when the investor-dealer market opens, all dealers are dealer-sellers as well as dealer-buyers. All this can be shown to imply that:

**Lemma 1** At where \( p = \beta (V_2^{LD} - V_1^{LD}) \), \( m_0^{SD} = m_2^{SD}. \)

**Proof.** Use superscripts \( SI \) and \( LI, \) respectively, for measures of small and large investors. Because all dealers are dealer-sellers as well as dealer-buyers and small investors trade with all dealers, in the steady state,

\[
n_B^{SI} \eta (\theta) = n_S^{SI} \eta (\theta),
\]

which implies that \( n_B^{SI} = n_S^{SI}. \) Because all small dealers enter the investor-dealer market holding a unit inventory and that they only trade with small investors,

\[
m_0^{SD} = \eta (\theta) \frac{n_B^{SI}}{n_D} n^{SD},
\]

\[
m_2^{SD} = \eta (\theta) \frac{n_S^{SI}}{n_D} n^{SD},
\]

from which we obtain \( m_0^{SD} = m_2^{SD}. \) ■

With \( m_0^{SD} = m_2^{SD} \), the buyers and sellers among small dealers in the inter-dealer market are equally numerous in this equilibrium so that they neither provide inventory nor capacity for large dealers.

**Case 4** \( p = \beta (V_2^{SD} - V_1^{SD}) \) with a fraction of \( S_2s \) selling in the inter-dealer market. This is the mirror opposite of case 2. Small dealers buy from more than they sell to large dealers, helping large dealers free up inventory capacities and providing immediacy for them on balance with \( p \) settling at a relatively low level as arising from a relatively large \( A. \)

**Case 5** \( p = \beta (V_2^{SD} - V_1^{SD}) \) with a fraction of \( S_1s \) buying in the inter-dealer market or \( p = \beta (V_3^{LD} - V_2^{LD}). \) The is the mirror opposite of case 1. Small dealers buy from large dealers only, providing immediacy for large dealers, with \( p \) settling at the lowest levels as arising from the largest \( A. \)
The above suggests that the result that small dealers provide immediacy should also generalize to where there are more than two inventory capacities, as similar mechanisms should be operative to give rise to smaller-capacity dealers selling (buying) the asset to (from) larger-capacity dealers when the asset supply is small (large).

8.4 Proofs

Proof of Proposition 1  It is useful to use the relation \( \eta(\theta) \frac{n_I^J}{n_I^J} = \mu(\theta) \frac{n_I^J}{n_I^J + n_B^J} \), for \( J = S, B \), for all the proofs in the following.

Then, given (3),

\[
V_{1SD} = W_{0SD} + p + \eta(\theta) \frac{n_I^B}{n_I^B} z_{IB} \frac{n_I^B}{2},
\]

and given (5),

\[
V_{0LD} = W_{1LD} - p + \eta(\theta) \frac{n_I^S}{n_I^S} z_{IS} \frac{n_I^S}{2},
\]

\[
V_{2LD} = W_{1LD} + p + \eta(\theta) \frac{n_I^B}{n_I^B} z_{IB} \frac{n_I^B}{2}.
\]

We can then calculate

\[
V_{1SD} - V_{0SD} = p + \eta(\theta) \frac{n_I^B}{n_I^B} z_{IB} \frac{n_I^B}{2} - \eta(\theta) \frac{n_I^S}{n_I^S} z_{IS} \frac{n_I^S}{2},
\]

\[
V_{1LD} - V_{0LD} = \eta(\theta) \frac{n_I^B}{n_I^B} z_{IB} \frac{n_I^B}{2} + p,
\]

\[
V_{2LD} - V_{1LD} = p - \eta(\theta) \frac{n_I^S}{n_I^S} z_{IS} \frac{n_I^S}{2},
\]

\[
(V_{1LD} - V_{0LD}) - (V_{1SD} - V_{0SD}) = \eta(\theta) \frac{n_I^S}{n_I^S} z_{IS} \frac{n_I^S}{2},
\]

\[
(V_{1SD} - V_{0SD}) - (V_{2LD} - V_{1LD}) = \eta(\theta) \frac{n_I^B}{n_I^B} z_{IB} \frac{n_I^B}{2}.
\]

Notice that the two surpluses \( z_{IB} \) and \( z_{IS} \) can at worst be equal to zero in equilibrium. This proves the Proposition.
**Proof of Proposition 2**

Substitute (6) into (9) and rearrange,

\[ U^I_S = \frac{\eta(\theta)n^D_B}{2n^{U}_B} p. \]  

(53)

Substitute the equation into (15) and rearrange,

\[ U^Q_H = \frac{\left(1 - \beta + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right) v + \beta \delta \frac{\eta(\theta)n^D_B}{2n^{U}_B} p}{(1 - \beta + \beta \delta) \left(1 - \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)}. \]

(54)

Substitute (10) into (13) and rearrange,

\[ U^I_B = \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta \left((1 - \delta) U^Q_H + \delta U^I_S\right). \]

(55)

Then, by (53), (54) and (55),

\[
(1 - \delta) (U^Q_H - U^I_B) + \delta U^I_S = \frac{(1 - \delta) \left(1 - \beta + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right) v + \left(\delta \frac{\eta(\theta)n^D_B}{2n^{U}_B} + \left(1 - \beta + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)(1 - \delta)\right) \frac{\eta(\theta)n^D_B}{2n^{U}_B} p}{(1 - \beta + \beta \delta) \left(1 - \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)\left(1 - \left(1 - \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)(1 - \delta)\beta\right)}.
\]

(56)

Set \( p = \beta \left(V^1 LD - V^0 LD\right) \) and by (49),

\[ p = \frac{\beta^{n^D_B}}{1 - \beta + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta} \left(\delta U^I_S + (1 - \delta) (U^Q_H - U^I_B)\right). \]

(57)

Then, use (56) to obtain

\[ p = \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta^2 (1 - \delta) \left(1 - \beta + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right) v
\times\frac{(1 - \beta + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta) \left(1 - \left(1 - \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)(1 - \beta)(1 - \delta)\right)}{(1 - \beta + \beta \delta) \left(1 - \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)\left(1 - \left(1 - \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta\right)(1 - \delta)\beta\right)}. \]

(58)

Given the positivity of \( p \) in (58) and by (53) and (57),

\[ 0 < \beta U^I_S < p < \beta \left(\delta U^I_S + (1 - \delta) (U^Q_H - U^I_B)\right), \]

(59)

from which it follows that \( z_{I_S} > 0 \) and \( z_{I_B} > 0 \).

Next, set \( p = \beta \left(V^1 SD - V^0 SD\right) \) and by (48),

\[ p = \frac{\beta^2 n^D_B}{2n^{U}_B} \beta^2 \left(1 - \delta\right) \left(U^Q_H - U^I_B\right) + \delta U^I_S + \frac{\eta(\theta)n^D_B}{2n^{U}_B} \beta^2 U^I_S. \]

(60)
Then, use (53) and (56) to obtain
\[
p = \frac{n_D}{{2n}_D^2} \beta^2 (1 - \delta) \left( 1 - \beta + \eta \frac{n_D}{2n}\right) v \left( \frac{1}{1 - \beta} \left( 1 - \frac{n_D}{2n}\right) \right)^v
\]
(61)

Given the positivity of \( p \) in (61) and by (53) and (60), the same conclusion as in (59) obtains.

Finally, set \( p = \beta \left( V^L_D - V^L_B \right) \) and by (50),
\[
p = \frac{\beta \eta \frac{n_D}{2n_D^2}}{1 - \beta + \eta \frac{n_D}{2n_D^2}} \beta U^S_I.
\]
(62)

With (53),
\[p = \beta U^S_I = 0,
\]
which also implies that \( z_{IS} = 0 \). Next, by (56),
\[
(1 - \delta) \left( U^N_H - U^I_B \right) + \delta U^I_S = \frac{(1 - \delta) v}{1 - \left( \frac{\eta \frac{n_D}{2n_D^2}}{1 - \delta} \right) (1 - \delta) \beta} > 0.
\]
(63)

And so,
\[0 = \beta U^I_S = p < \beta \left( U^N_H - U^I_B \right) + \delta U^I_S,
\]
which implies that \( z_{IB} > 0 \).

**Proof of Proposition 3** Solve (23)-(25) for
\[
n^Q_N = \frac{(1 - \delta) \eta \frac{n_D}{n^D} e}{\delta + (1 - \delta) \eta \frac{n_D}{n^D} \delta},
\]
(64)

\[
n^I_S = \frac{n^D_S e}{\delta + (1 - \delta) \eta \frac{n_D}{n^D}},
\]
(65)

\[
n^I_B = \frac{e}{\delta + (1 - \delta) \eta \frac{n_D}{n^D}}.
\]
(66)

Substituting the above into (1) and (22) gives, respectively,
\[
\theta = \left( \delta + (1 - \delta) \eta \frac{n_D}{n^D} \right) \frac{n^D_B}{n_B + n^D} \frac{n^D}{n^D _1 + 2n^D_B} = \frac{1}{\delta + (1 - \delta) \eta \frac{n_D}{n^D}} \frac{n^D_S e + n^D SD + n^L_D + 2n^L_D}{A}.
\]
(68)
**Selling Eq.** In the Selling Eq., by Table 1, \( n_0^{SD} = n^{SD} \), \( n_1^{SD} = 0 \), and \( n_2^{LD} = 0 \). Then, \( n_S^D = n_1^{LD} \) and \( n_B^D = n^D \), in which case (67) and (68) specialize to, respectively,

\[
\theta = \left( \delta + (1 - \delta) \eta(\theta) \frac{n_1^{LD}}{n^D} \right) \frac{n^D}{n_1^{LD} + n^D} \frac{n^D}{e}. \tag{69}
\]

\[
\frac{1 - \delta}{\delta} \eta(\theta) + 1 \frac{n_1^{LD}}{n_1^{LD} + n^D} e + n_1^{LD} = A. \tag{70}
\]

Solve (70) for

\[
\eta(\theta) = \frac{\left( A - n_1^{LD} \right) \delta - \frac{n_1^{LD}}{n^D} e}{\frac{e}{\delta} + n_1^{LD} - A} \left( 1 - \delta \right) \frac{n_1^{LD}}{n^D}. \tag{71}
\]

Evaluate both sides of (69) by means of \( \eta \), equate the RHS of the resulting expression to the RHS of (71) and then use \( \mu = \eta(\theta) / \theta \) to obtain

\[
\delta \frac{A - n_1^{LD} - \frac{n_1^{LD}}{n^D} e}{n_1^{LD} (n^D - n_1^{LD})} \left( n_1^{LD} + n^D \right) = \mu \left( \frac{n^D - n_1^{LD}}{\frac{e}{\delta} + n_1^{LD} - A} \frac{n^D}{n_1^{LD} + n^D} \right). \tag{72}
\]

Any solution for \( n_1^{LD} \in [0, n^{LD}] \) to (72) is a Selling Eq.

The RHS of (72) is well-defined and increasing for \( n_1^{LD} \in \left[ \max \left\{ A - \frac{e}{\delta}, 0 \right\}, n^{LD} \right] \) whereas the LHS is decreasing over the same range if

\[
\left( A - \frac{e}{\delta} - n^D \right) n_1^{LD} (n_1^{LD} + n^D) - \left( A - n_1^{LD} - \frac{n_1^{LD} e}{n^D} \right) n^D (n^D - n_1^{LD}) < 0,
\]

which is guaranteed to hold under

\[
n^{LD} > A - \frac{e}{\delta}, \tag{73}
\]

the same condition for the interval \( \left[ \max \left\{ A - \frac{e}{\delta}, 0 \right\}, n^{LD} \right] \) to be non-empty.

Hence, a unique solution \( n_1^{LD} \in \left[ \max \left\{ A - \frac{e}{\delta}, 0 \right\}, n^{LD} \right] \) to (72) exists if and only if:

1. The RHS of (72) is not smaller than the LHS at \( n_1^{LD} = n^{LD} \); i.e.,

\[
\frac{\delta}{1 - \delta} \left( A - n^{LD} - \frac{n^{LD} e}{n^D} \right) \left( n^{LD} + n^D \right) \leq \mu \left( \frac{n^D - n^{LD}}{\frac{e}{\delta} + n^{LD} - A} \frac{n^D}{n^{LD} + n^D} \right). \tag{74}
\]

2. The RHS of (72) is not larger than the LHS at \( n_1^{LD} = \max \left\{ A - \frac{e}{\delta}, 0 \right\} \). One can check that the condition is met.

Notice that (73) and (74) are the two conditions in Proposition 3(a), and that (72) just yields \( n_1^{LD} = n^{LD} \) at where (74) holds as an equality.

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**Balanced Eq.** In the Balanced Eq., by Table 1, \( n_1^{LD} = n^{LD} \), while \( n_0^{LD} = n_2^{LD} = 0 \). Then, \( n_S^D = n_1^{SD} + n^{LD} \) and \( n_B^D = n^D - n_1^{SD} \), in which case (67) and (68) specialize to, respectively,

\[
\theta = \left( \delta + \eta(\theta) \frac{n_1^{SD} + n^{LD}}{n^D} (1 - \delta) \right) \frac{n^D - n_1^{SD} n^D}{n^D + n^{LD}} e^{1 - \delta},
\]

\[
\frac{1 - \delta}{\delta + \eta(\theta) \frac{n_1^{SD} + n^{LD}}{n^D}} + \frac{n_1^{SD} + n^{LD}}{n^D - n_1^{SD}} e^{1 - \delta} + n_1^{SD} + n^{LD} = A.
\]

Solve (76) for

\[
\eta(\theta) = \frac{A \delta - (n_1^{SD} + n^{LD}) \delta - \frac{n_1^{SD} + n^{LD}}{n^D - n_1^{SD}} e^{1 - \delta}}{(\frac{e}{\delta} + n_1^{SD} + n^{LD} - A) (1 - \delta) \frac{n_1^{SD} + n^{LD}}{n^D}}.
\]

Evaluate both sides of (75) by means of \( \eta \), equate the RHS of the resulting expression to the RHS of (77) and then use \( \mu = \eta(\theta) / \theta \) to obtain

\[
\delta \left( A - n_1^{SD} - n^{LD} - \frac{n_1^{SD} + n^{LD}}{n^D - n_1^{SD}} e^{1 - \delta} \right) \left( n^D - n_1^{SD} \right) = \mu \left( \frac{n^D - 2n_1^{SD}}{\frac{e}{\delta} + n_1^{SD} + n^{LD} - A n^D + n^{LD}} \right).
\]

Any solution for \( n_1^{SD} \in [0, n^{SD}] \) to (78) is a Balanced Eq.

Consider where

\[
A - n^{LD} - \frac{e}{\delta} - \frac{n^{SD}}{2} \leq 0.
\]

Then, the RHS of (78) is defined for \( n_1^{SD} \in \left[ \max \{ A - n^{LD} - \frac{e}{\delta}, 0 \}, \frac{n^{SD}}{2} \right] \) over which it is positive and increasing in \( n_1^{SD} \). By (79), the LHS is equal to zero at \( n_1^{SD} = \tilde{n}_1^{SD} \) for some \( \tilde{n}_1^{SD} \leq \frac{n^{SD}}{2} \), satisfying,

\[
A - \tilde{n}_1^{SD} - n^{LD} - \tilde{n}_1^{SD} + n^{LD} e^{1 - \delta} = 0,
\]

and falls below zero for any larger \( n_1^{SD} \). This means that the admissible \( n_1^{SD} \) is restricted to the interval \( \left[ \max \{ A - n^{LD} - \frac{e}{\delta}, 0 \}, \frac{n^{SD}}{2} \right] \). In particular, at \( n_1^{SD} = \frac{n^{SD}}{2} \), the RHS of (78), which is positive, must exceed the LHS, which is equal to 0. Furthermore, we show in the Online Appendix that the LHS is monotone decreasing over where it is positive. Hence, a unique solution \( n_1^{SD} \in \left[ \max \{ A - n^{LD} - \frac{e}{\delta}, 0 \}, \frac{n^{SD}}{2} \right] \) to (78) exists if and only if the RHS of (78) is not larger than the LHS at \( n_1^{SD} = \max \{ A - n^{LD} - \frac{e}{\delta}, 0 \} \).

For \( A - n^{LD} - \frac{e}{\delta} < 0 \) so that the lower bound \( n_1^{SD} = 0 \), the condition is that of (74) holding in reverse. This proves the first part of Proposition 3(b).
For $A - n^{LD} - \frac{\xi}{\delta} \geq 0$, which when combined with (79) restricts $A - \frac{\xi}{\delta} \in \left[ n^{LD}, n^{LD} + \frac{n^{SD}}{2} \right]$, which constitutes the condition in the second part of Proposition 3(b). At the lower bound $n_1^{SD} = A - n^{LD} - \frac{\xi}{\delta}$, the RHS is equal to 0. Then, the condition that the LHS of (78) is not smaller than the RHS is met at the lower bound if the LHS is non-negative; i.e.,

$$1 + \frac{1}{1 - \frac{n^{LD} + n^{D}}{A - \frac{\xi}{\delta}}} \geq 0,$$

which holds for

$$n^{LD} + n^{D} \leq A - \frac{\xi}{\delta}$$

or

$$\frac{n^{LD} + n^{D}}{2} \geq A - \frac{\xi}{\delta}.$$

The first case is ruled out by (79) and so only the second case is relevant, which can be seen to be identical to (79). Notice in case (79) holds as an equality, the two sides of (78) are equal to 0 at $n_1^{SD} = A - n^{LD} - \frac{\xi}{\delta} = \frac{n^{SD}}{2}$. This completes the proof of Proposition 3(b).

Rewrite (78) as

$$\frac{\delta}{1 - \delta} \frac{n_1^{SD} + n^{LD} + \frac{n^{SD} + n^{LD}}{n^{SD} - n^{LD}} \delta - A}{(n_1^{SD} + n^{LD})(2n_1^{SD} - n^{SD})} (n^{D} + n^{LD}) = \mu \left( \frac{2n_1^{SD} - n^{SD}}{A - \frac{\xi}{\delta} - n_1^{SD} - n^{LD}} \frac{n^{D}}{n^{D} + n^{LD}} \right). \quad (81)$$

Consider where

$$A - n^{LD} - \frac{\xi}{\delta} - \frac{n^{SD}}{2} > 0. \quad (82)$$

Then, the RHS of (81) is defined for $n_1^{SD} \in \left[ \frac{n^{SD}}{2}, \min \left\{ A - \frac{\xi}{\delta} - n^{LD}, n^{SD} \right\} \right]$ over which it is positive but decreasing in $n_1^{SD}$. By (82), the LHS is equal to zero at $n_1^{SD} = \tilde{n}_1^{SD}$ for $\tilde{n}_1^{SD} \geq \frac{n^{SD}}{2}$ defined in (80) and falls below zero for any smaller $n_1^{SD}$. This means that the admissible $n_1^{SD}$ is restricted to the interval $[\tilde{n}_1^{SD}, \min \left\{ A - \frac{\xi}{\delta} - n^{LD}, n^{SD} \right\}]$. In particular, at $n_1^{SD} = \tilde{n}_1^{SD}$, the RHS of (81), which is positive, must exceed the LHS, which is equal to 0. Furthermore, we show in the Online Appendix that the LHS is monotone increasing over where it is positive. Hence, a unique solution $n_1^{SD} \in \left[ \tilde{n}_1^{SD}, \min \left\{ A - \frac{\xi}{\delta} - n^{LD}, n^{SD} \right\} \right]$ to (81) exists if and only if the RHS of (81) is not larger than the LHS at $n_1^{SD} = \min \left\{ A - \frac{\xi}{\delta} - n^{LD}, n^{SD} \right\}$.

For $A - \frac{\xi}{\delta} - n^{LD} < n^{SD}$, which when combined with (82) restricts $A - \frac{\xi}{\delta} \in \left( n^{LD} + \frac{n^{SD}}{2}, n^{D} \right)$, which constitutes the condition in the first part of Proposition 3(c). At the upper bound $n_1^{SD} = A - \frac{\xi}{\delta} - n^{LD}$, the RHS is equal to 0 while the LHS is proportional to $\frac{2(A - \frac{\xi}{\delta} - n^{LD} - \frac{n^{SD}}{2})}{n^{D} + n^{SD} - A + \frac{\xi}{\delta}} > 0$. 

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For $A - \delta - n^{LD} \geq n^{SD}$ so that the upper bound is $n_1^{SD} = n^{SD}$, the condition that RHS of (81) is not larger than the LHS is

$$
\frac{\delta}{1 - \delta} \frac{n^D + \frac{n^D}{n^{LD}} - A (n^D + n^{LD})}{n^{SD}} \geq \mu \left( \frac{n^{SD}}{A - \frac{e}{\delta} - n^{LD}} \frac{n^D}{n^D + n^{LD}} \right),
$$

(83)
which is the condition in the second part of Proposition 3(c). Notice that at where the above just holds as an equality, (81) yields $n_1^{SD} = n^{SD}$.

**Buying Eq.** In the Buying Eq., by Table 1, $n_1^{SD} = n^{SD}$, $n_0^{SD} = 0$, and $n_0^{LD} = 0$. Then, $n_0^D = n^D$ and $n_0^D = n_1^D$, in which case (67) and (68) specialize to, respectively,

$$
\theta = (\delta + (1 - \delta) \eta (\theta)) \frac{n_1^{LD}}{n_1^D + n^D} e,
$$

(84)
$$
\frac{1 - \delta}{\delta + (1 - \delta) \eta (\theta)} \frac{n^D}{n_1^{LD}} e + n^D + n^{LD} - n_1^{LD} = A.
$$

(85)

Solve (85) for

$$
\eta (\theta) = \frac{(A - (n^D + n^{LD} - n_1^{LD})) \delta - n_1^{LD} e}{(\frac{e}{\delta} + (n^D + n^{LD} - n_1^{LD}) - A) (1 - \delta)}.
$$

(86)
Evaluate both sides of (84) by means of $\eta$, equate the RHS of the resulting expression to the RHS of (86) and then use $\mu = \eta (\theta) / \theta$ to obtain

$$
\frac{\delta}{1 - \delta} \frac{(n^D - n_1^{LD}) n^D}{(n^D - n_1^{LD}) n^D} = \mu \left( \frac{n^D - n_1^{LD}}{A - n^D - n^{LD} + n_1^{LD} - \frac{e}{\delta} n_1^{LD} + n^D} \right).
$$

(87)

Any solution for $n_1^{LD} \in [0, n^{LD}]$ to (87) is a Buying Eq..

The RHS of (87) is defined for $n_1^{LD} \in \left[ \max \left\{ 0, -A + n^D + n^{LD} + \frac{e}{\delta} \right\}, n^{LD} \right]$, over which it is increasing in $n_1^{LD}$. The interval is guaranteed non-empty where

$$
-A + \frac{e}{\delta} + n^D \leq 0.
$$

(88)

The derivative of the LHS has the same sign as

$$
-A + n^D + n^{LD} - n_1^{LD} + \frac{e}{\delta} \left( \frac{n^D}{n_1^{LD}} - \frac{1}{2} \left( \frac{n^D}{n_1^{LD}} \right)^2 + \frac{1}{2} \right) - \frac{(n^D)^2 - (n_1^{LD})^2}{2n^D} < -A + n^D + n^{LD} - n_1^{LD} + \frac{e}{\delta},
$$

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where the inequality is due to $\frac{n^D}{n^D} - \frac{1}{2}(\frac{n^D}{n^D})^2 + \frac{1}{2} \leq 1$. Then, the LHS is decreasing in $n_1^{LD}$ over where the RHS is defined.

Hence, a unique solution $n_1^{LD} \in [\max \{0, -A + n^D + n^{LD} + \frac{\varepsilon}{\bar{y}}\}, n^{LD}]$ to (87) exists if and only if:

1. The RHS of (87) is not larger than the LHS at $n_1^{LD} = \max \{0, -A + n^D + n^{LD} + \frac{\varepsilon}{\bar{y}}\}$.
   One can check that the condition is met.

2. The RHS of (87) is not smaller than the LHS at $n_1^{LD} = n^{LD}$; i.e., where (83) holds in reverse.

   Notice that (87) yields $n_1^{LD} = n^{LD}$ (and so $n_2^{LD} = 0$) just at where (83) holds as an equality. This proves Proposition 3(d).

**Market clearing** We turn next to verify that the solutions for $n_1^{LD}$ to (72) for the Selling Eq., $n_1^{SD}$ to (78) for the Balanced Eq., and $n_1^{LD}$ to (87) for the Buying Eq. give rise to allocations that indeed clear the inter-dealer market for the three types of equilibrium, respectively.

In the Selling Eq., the market clearing condition (16), by means of (31), (32), and (34), reduces to

$$n^{LD} - n_1^{LD} \geq 0$$

that obviously must hold. In the Buying Eq., the market clearing condition (19), by means of (26), (32), and (34), reduces to exactly the same condition.

In the Balanced-Selling Eq., the market clearing condition (17), by means of (31), (32), and (34), reduces to

$$n_1^{SD} \geq 0,$$

whereas in Balanced-Buying Eq., the market clearing condition (18), by means of (26), (32), and (34), reduces to

$$n^{SD} - n_1^{SD} \geq 0.$$
Both conditions are guaranteed to hold. Furthermore, the condition \( m_{LD}^0 \geq m_{LD}^2 \), by means of (32), (34), (65), (66) and the definitions for \( n_{BD}^D \) and \( n_{DS}^D \), can be shown to be equivalent to

\[ n_B^I \geq n_S^I \iff n_B^D \geq n_S^D \iff n_1^SD \leq \frac{n_{SD}^I}{2}, \]

as presented in Proposition 3(b).

**Proof of Corollary 1** The Corollary follows directly from Proposition 3 and the characteristics of the three equilibria in Table 1.

**Proof of Proposition 4** From (65) and (66), \( \frac{n_{BD}^D}{n_B^I} < 1 \) is equivalent to \( n_S^I < n_B^I \). In the Selling Equilibrium and the Balanced-Selling Equilibrium, \( \frac{n_{BD}^D}{n_B^I} < 1 \), while in the Balanced-Buying Equilibrium and the Buying Equilibrium, \( \frac{n_{BD}^D}{n_B^I} > 1 \). In equilibrium, a dealer meets an investor-seller at probability \( \mu(\theta) \frac{n_I^S}{n_I^S + n_I^B} \) and an investor-buyer at probability \( \mu(\theta) \frac{n_I^B}{n_I^S + n_I^B} \). The former probability is larger than the latter if and only if \( n_I^S > n_I^B \).

**Proofs of Proposition 5 and Corollary 2** Use \( n_{LD}^I = \gamma n_{LD}^D \) and \( n_{SD}^I = (1 - \gamma) n_D^D \), the four bounds for \( A \) in Proposition 3 can be rewritten as four bounds for \( n_D^D \) as in, respectively,

\[ A > n_D^D + \frac{e}{\delta} \Rightarrow n_D^D < A - \frac{e}{\delta}, \quad (89) \]

\[ \frac{\gamma + 1}{2} n_D^D + \frac{e}{\delta} < A \leq n_D^D + \frac{e}{\delta} \Rightarrow A - \frac{e}{\delta} \leq n_D^D < \frac{2}{1 + \gamma} \left( A - \frac{e}{\delta} \right), \quad (90) \]

\[ \gamma n_D^D + \frac{e}{\delta} \leq A < \frac{\gamma + 1}{2} n_D^D + \frac{e}{\delta} \Rightarrow \frac{2}{1 + \gamma} \left( A - \frac{e}{\delta} \right) < n_D^D \leq \frac{1}{\gamma} \left( A - \frac{e}{\delta} \right), \quad (91) \]

\[ A \in \left[ 0, \gamma n_D^D + \frac{e}{\delta} \right) \Rightarrow n_D^D > \frac{1}{\gamma} \left( A - \frac{e}{\delta} \right), \quad (92) \]

and then \( \Omega_S \) and \( \Omega_B \) can be expressed as functions of \( n_D^D \) as in, respectively,

\[ \Omega_S \left( n_D^D \right) = \frac{\delta}{1 - \delta} \frac{\gamma + 1}{1 - \gamma} A - \frac{n_D^D - \frac{e}{\delta}}{A - \frac{e}{\delta}} - \mu \left( \frac{n_D^D}{A - \frac{e}{\delta}} - 1 \right), \quad (93) \]

\[ \Omega_B \left( n_D^D \right) = \frac{\delta}{1 - \delta} \frac{\gamma + 1}{1 - \gamma} n_D^D + \frac{1}{\gamma} \frac{e}{\delta} - A \left( \frac{n_D^D}{A - \frac{e}{\delta}} - 1 \right) - \mu \left( \frac{n_D^D}{A - \frac{e}{\delta}} - 1 \right). \quad (94) \]

The first three intervals (89)-(91) are non-empty only for \( A > \frac{e}{\delta} \). If in addition, \( A > \frac{1}{\gamma} \frac{e}{\delta} \),

\[ \lim_{n_D^D \to 0} \Omega_B = -\infty, \quad \lim_{n_D^D \to \frac{e}{\delta}} \Omega_B > 0, \quad \frac{\partial \Omega_B}{\partial n_D^D} > 0. \]
Thus, by Proposition 3(d) and (89), starting at \( n^D = 0 \), the Buying Eq. holds and it holds until \( n^D \) reaches \( n_{BE}^D < A - \frac{\varepsilon}{\delta} \) as defined in (27) at which point \( \Omega_B = 0 \). Thereafter, by Proposition 3(c), the Balanced-Buying Eq. first holds and by (90), it holds as long as \( n^D < \frac{2}{1+\gamma} (A - \frac{\varepsilon}{\delta}) \). Then, by (91) and Proposition 3(b), at \( n^D = \frac{2}{1+\gamma} (A - \frac{\varepsilon}{\delta}) \), \( n^{SD} = n^{SD}/2 \) and that the Balanced-Selling Eq. holds for larger \( n^D \). Since, where \( A > \frac{\varepsilon}{\delta} \),

\[
\lim_{n^D \to \frac{1}{\gamma} (A - \frac{\varepsilon}{\delta})} \Omega_S > 0 \quad \lim_{n^D \to \infty} \Omega_S < 0, \quad \frac{\partial \Omega_S}{\partial n^D} < 0,
\]

the Balanced-Selling Eq. continues to hold until \( n^D \) reaches \( n_{SE}^D > \frac{1}{\gamma} (A - \frac{\varepsilon}{\delta}) \) as defined in (28) at which point \( \Omega_S = 0 \). Thereafter, the Selling Eq. holds. This completes the proof of Corollary 2.

We shall prove Proposition 5 by showing that \( A^D \) is increasing in \( n^D \) in each type of equilibrium.

**Buying Eq.** In the Buying Eq., \( n_1^{SD} = n^{SD} \) and \( n_0^{LD} = 0 \), so that

\[
A^D = n^{SD} + n_1^{LD} + 2 \left( n^{LD} - n_1^{LD} \right) = (1 + \gamma) n^D - n_1^{LD}.
\]

This yields,

\[
n_1^{LD} = (1 + \gamma) n^D - A^D. \tag{95}
\]

Then, the equation for \( n_1^{LD} \) in the Buying Eq. (87) can be turned into an equation for \( A^D \),

\[
\frac{\delta}{1-\delta} \left( A^D - A + \frac{n^D}{(1+\gamma)n^D - A} \right) \left( (2 + \gamma) n^D - A^D \right) - \mu \left( \frac{A^D - \gamma n^D}{A - A^D - \frac{\varepsilon}{\delta}} \right) \left( 2 + \gamma \right) n^D - A^D \right) = 0. \tag{96}
\]

Because in the Buying Eq. \( n_1^{LD} \in \left[ \max \left\{ 0, -A + n^D + \gamma n^D + \frac{\varepsilon}{\delta} \right\}, \gamma n^D \right] \), by (95), the interval for

\[
A^D \in \left[ n^D, \min \left\{ (1 + \gamma) n^D, A - \frac{\varepsilon}{\delta} \right\} \right]. \tag{97}
\]

We have shown that the LHS of (96) is decreasing in \( n_1^{LD} \) in the proof of Proposition 3. Then, it must be increasing in \( A^D \) because there is a one-to-one and negative relation between \( A^D \) and \( n_1^{LD} \). To show that \( \partial A^D / \partial n^D > 0 \), we need to show that the LHS is decreasing in
Differentiating with respect to \( n^D \), substituting from (96), and using \( \mu(\theta) = \frac{\eta(\theta)}{\theta} \) and \( \mu(\theta) + \mu'(\theta) \theta = \eta'(\theta) \), it can be shown that the condition is

\[
\eta' \left( \frac{A^D - \gamma n^D}{A - A^D - \gamma} \right) \frac{A^D - \gamma n^D}{A - A^D - \gamma} \frac{n^D}{A - A^D - \gamma n^D} < \frac{\eta' \left( \frac{A^D - \gamma n^D}{A - A^D - \gamma} \right)}{\left( (1 + \gamma) n^D - A^D \right) n^D}. \tag{98}
\]

Given \( \frac{\epsilon}{\delta} \leq A - A^D \) by (97), the RHS can be shown to be greater than one. Since the elasticity \( \theta \eta'(\theta) / \eta(\theta) \) is restricted to be less than unitary to be begin with given the concavity of the \( \eta(\theta) \) function, (98) is guaranteed to hold.

**Balanced-Buying Eq.** In the Balanced Eq., \( n_1^{LD} = n^D \), so that

\[
A^D = n_1^{SD} + n^{LD} = n_1^{SD} + \gamma n^D.
\]

This yields,

\[
n_1^{SD} = A^D - \gamma n^D. \tag{99}
\]

Then, the equation for \( n_1^{SD} \) in the Balanced-Buying Eq. (81) can be turned into an equation for \( A^D \),

\[
\frac{\delta}{1 - \delta} \frac{A^D + \frac{\epsilon}{(1 + \gamma) n^D - A^D}}{A^D - \frac{1}{2}} - A^D (2 A^D - (\gamma + 1) n^D) n^D (1 + \gamma) - \mu \left( \frac{2 A^D - (\gamma + 1) n^D}{A - \frac{\epsilon}{\delta} - A^D} \frac{1}{1 + \gamma} \right) = 0. \tag{100}
\]

From (80) and (99), define \( \tilde{A}^D \) as satisfying,

\[
A - \tilde{A}^D - \frac{\tilde{A}^D}{(1 + \gamma) n^D - \tilde{A}^D} \frac{\epsilon}{\delta} = 0,
\]

and that, where \( n_1^{SD} \leq \frac{n^{SD}}{2} \),

\[
\tilde{A}^D \geq \left( \frac{1 + \gamma}{2} \right) n^D. \tag{101}
\]

This Balanced Eq. is defined over \( n_1^{SD} \in \left[ n_1^{SD}, \min \{ A - \frac{\epsilon}{\delta} - n^{LD}, n^{SD} \} \right] \). In terms of \( A^D \), the bounds are

\[
A^D \in \left[ \tilde{A}^D, \min \{ A - \frac{\epsilon}{\delta}, n^D \} \right]. \tag{102}
\]

We have shown that in the proof of Proposition 3, the LHS of (100) is increasing in \( n_1^{SD} \) and therefore in \( A^D \) over where the equation is satisfied. To show that \( \partial A^D / \partial n^D > 0 \), we need to
show that the LHS is decreasing in \( n^D \). Differentiating (100) with respect to \( n^D \), substituting from (100) and then using \( \mu(\theta) = \frac{n(\theta)}{\theta} \) and \( \theta \mu'(\theta) + \mu(\theta) = \eta'(\theta) \), it can be shown that the condition is

\[
\eta' \left( \frac{2A^D - (\gamma+1)n^D}{A - \frac{\gamma}{\delta} - A^D} \right) \frac{1}{1 + \gamma} \geq \frac{(1+\gamma)A^D}{(1+\gamma)n^D - A^D} - \frac{1}{n^D} \left( \frac{2A^D - (\gamma+1)n^D}{1 + \gamma} \right).
\]

The RHS is greater than 1 if

\[
(2A^D - (\gamma+1)n^D) - (1 + \gamma) n^D \left( (1 + \gamma) n^D - A^D \right) \frac{e}{\delta} + 2 (A - A^D) \left( (1 + \gamma) n^D - A^D \right)^2 > 0. \tag{103}
\]

If the first line is positive, then the whole expression is positive for sure. Suppose not. By (102); i.e., \( A^D \leq A - \frac{\gamma}{\delta} \), then (103) is greater than an expression proportional to

\[4 (A^D)^2 - 4A^D n^D - 4A^D n^D \gamma + (n^D)^2 \gamma^2 + 2 (n^D)^2 \gamma + (n^D)^2.\]

Differentiate with respect to \( A^D \),

\[
8A^D - 4n^D - 4n^D \gamma = 8 \left( A^D - \frac{(1 + \gamma) n^D}{2} \right) > 0 \text{ by (101) and (102)}.
\]

Thus, (103) is increasing in \( A^D \), for \( A^D \geq \tilde{A}^D \geq \left( \frac{1+\gamma}{2} \right) n^D \). Then, it must be weakly greater than

\[
4 \left( \frac{1 + \gamma}{2} \right) n^D)^2 - 4 \left( \frac{1 + \gamma}{2} \right) (n^D)^2 - 4 \left( \frac{1 + \gamma}{2} \right) (n^D)^2 \gamma + (n^D)^2 \gamma^2 + 2 (n^D)^2 \gamma + (n^D)^2 = 0.
\]

**Balanced-Selling Eq.** Given (99), the equation for \( n_1^{SD} \) in the Balanced-Selling Eq. (78) can be turned into an equation for \( A^D \),

\[
\frac{\delta}{1 - \delta} \frac{A - A^D - \frac{A^D}{(1+\gamma)n^D - 2A^D} \frac{\gamma}{\delta} (1 + \gamma) n^D}{A^D} - \mu \left( \frac{(1 + \gamma)n^D - 2A^D}{\frac{\gamma}{\delta} + A^D - A} \right) = 0. \tag{104}
\]

This Balanced Eq. is defined over \( n_1^{SD} \in [\max \left\{ A - \frac{\gamma}{\delta} - n^{LD} , 0 \right\} , \tilde{n}_1^{SD}] \). In terms of \( A^D \), the bounds are

\[
A^D \in \left[ \max \left\{ A - \frac{e}{\delta} , n^D \right\} , \tilde{A}^D \right]. \tag{105}
\]
We have shown that, in the proof of Proposition 3, the LHS of (104) is decreasing in $n_{1}^{SD}$ and therefore in $A^{D}$ over where the equation is satisfied. To show that $\partial A^{D}/\partial n^{D} > 0$, we need to show that the LHS is increasing in $n^{D}$. Differentiating (104) with respect to $n^{D}$, substituting from (104) and then using $\mu(\theta) = \frac{\eta(\theta)}{\theta}$ and $\theta \mu'(\theta) + \mu(\theta) = \eta'(\theta)$, it can be shown that the condition is

$$\frac{\eta'\left(\frac{(1+\gamma)n^{D}-2A^{D}}{\frac{\delta}{\tau}+A^{D}-A} - \frac{1}{1+\gamma}\right)}{\eta\left(\frac{(1+\gamma)n^{D}-2A^{D}}{\frac{\delta}{\tau}+A^{D}-A} - \frac{1}{1+\gamma}\right)} < \left(\frac{(1+\gamma)A^{D}}{(1+\gamma)n^{D}-A^{D}}\right)^{\frac{\epsilon}{\delta}} \frac{(A - A^{D} - \frac{A^{D}}{\frac{\delta}{\tau}+A^{D}-A} n^{D})}{(\gamma + 1)n^{D} - 2A^{D}}. \quad (106)$$

From (105) that $\frac{\delta}{\tau} \geq A - A^{D}$, the RHS of (106) is greater than

$$\frac{1}{(1+\gamma)n^{D}} \left(1+\gamma\right)n^{D} - 2A^{D} + \frac{(1+\gamma)n^{D}A^{D}}{(1+\gamma)n^{D} - A^{D}} \geq \frac{4 + 2\sqrt{2}}{4 + 3\sqrt{2}} \approx 0.828, \quad (107)$$

given that the first term above is minimized at $A^{D} = \frac{1+\gamma}{2+\sqrt{2}}n^{D}$. Thus, (106) is guaranteed to hold for $\theta \eta'(\theta)/\theta \leq 0.828$ – the condition in the Proposition.

**Selling Eq.** In the Selling Eq., $n_{1}^{SD} = n_{2}^{LD} = 0$, so that $A^{D} = n_{1}^{LD}$. Then, the equation for $n_{1}^{LD}$ in the Selling Eq. (72) turns into an equation for $A^{D}$,

$$\frac{\delta}{1-\delta} \left(\frac{A - A^{D} - \frac{A^{D}}{\frac{\delta}{\tau}+A^{D}-A} n^{D}}{A^{D} - A^{D}}\right) - \mu\left(\frac{\frac{\epsilon}{\tau}+A^{D} - A A^{D} + n^{D}}{A^{D} - A^{D}}\right) = 0. \quad (108)$$

Since in the Selling Eq., $n_{1}^{LD} \in \left[\max\left\{A - \frac{\epsilon}{\tau}, 0\right\}, n^{LD}\right]$, there is the same range for

$$A^{D} \in \left[\max\left\{A - \frac{\epsilon}{\tau}, 0\right\}, \gamma n^{D}\right]. \quad (109)$$

The proof of Proposition 3 shows that the LHS of (108) is decreasing in $n_{1}^{LD} = A^{D}$. To show that $\partial A^{D}/\partial n^{D} > 0$, we need to show that the LHS of (108) is increasing in $n^{D}$. Differentiating (108) with respect to $n^{D}$, substituting from (108) and then using $\mu(\theta) = \frac{\eta(\theta)}{\theta}$ and $\theta \mu'(\theta) + \mu(\theta) = \eta'(\theta)$, it can be shown that the condition is

$$\frac{\eta'\left(\frac{n^{D}-A^{D}}{\frac{\delta}{\tau}+A^{D}-A A^{D}+n^{D}}\right)}{\eta\left(\frac{n^{D}-A^{D}}{\frac{\delta}{\tau}+A^{D}-A A^{D}+n^{D}}\right)} < \frac{A - A^{D}}{A^{D} - A^{D}} - \frac{n^{D}}{\frac{\delta}{\tau}+A^{D} - A A^{D}+n^{D}}. \quad (110)$$

From (109) that $A^{D} \geq \max\left\{A - \frac{\epsilon}{\tau}, 0\right\}$, the RHS of (110) is greater than

$$\frac{n^{D}}{n^{D} + \frac{A^{D}(n^{D} - A^{D})}{(A^{D} + n^{D})}} > \frac{1}{4} \frac{\sqrt{2}}{\sqrt{2} - 1} \approx 0.854.$$
given that the first term above is minimized at \( A^D = (\sqrt{2} - 1) n^D \). Thus, (110) is guaranteed to hold for \( \theta \eta' (\theta) / \theta \leq 0.854 \), which is subsumed by the condition in the Proposition.

Proofs of Propositions 6 and 7

By means of (65) and then via (67), we obtain two equations for

\[
\eta (\theta) n_I^L = \frac{\eta (\theta) n^D}{\delta (1 - \delta) \eta (\theta) n^D/n_D}, \tag{111}
\]

\[
\eta (\theta) n_I^S = \mu (\theta) \frac{n^D n_D}{n_B^D + n_S^D}. \tag{112}
\]

By means of (66) and then via (67), we obtain two equations for

\[
\eta (\theta) n_B^I = \frac{\eta (\theta) e}{\delta (1 - \delta) \eta (\theta) n^D/n_D}, \tag{113}
\]

\[
\eta (\theta) n_B^L = \mu (\theta) \frac{n_B^D n^D}{n_B^D + n_S^D}. \tag{114}
\]

Selling Eq. In the Selling Eq., \( n_{SD}^I = n_{LD}^2 = 0 \) and \( n_{SD}^0 = n_{SD} \) and so \( n_B^D = n^D \) and \( n_S^D = n_{1 LD}^D \). In this case, by means of (31) and (34), trading volume,

\[ TV = m_{1 SD}^I + m_{2 LD}^I = \eta (\theta) \frac{n_I^L}{n^D} (n_{1 LD}^D + n_{SD}), \]

whereas, (111) and (112) specialize to, respectively,

\[ \eta (\theta) n_I^L = \frac{e \eta (\theta) n_{1 LD}^I}{\delta (1 - \delta) \eta (\theta) n_{1 LD}^I} = L_{SE}^S (\theta; n_{1 LD}^D), \]

\[ \eta (\theta) n_I^S = \mu (\theta) \frac{n_{1 LD}^D n_{1 LD}}{n_B^D + n_{1 LD}} = R_{SE}^S (\theta; n_{1 LD}^D). \]

Set \( n_{1 LD}^D \) equal to the equilibrium value. Equilibrium \( \theta \) satisfies \( L_{SE}^S (\theta; n_{1 LD}^D) = R_{SE}^S (\theta; n_{1 LD}^D) \).

Note that \( L_{SE}^S (\theta; n_{1 LD}^D) \) is increasing and \( R_{SE}^S (\theta; n_{1 LD}^D) \) is decreasing in \( \theta \), and that both \( L_{SE}^S (\theta; n_{1 LD}^D) \) and \( R_{SE}^S (\theta; n_{1 LD}^D) \) are increasing in \( A \), where \( n_{1 LD}^D \) is increasing in \( A \) in the Selling Eq. as implied by (72). All together then, \( \eta (\theta) n_I^L \) is increasing in \( A \) and so does \( TV \).

On the other hand, from (113) and (114), respectively, where \( n_B^D = n^D \) and \( n_S^D = n_{1 LD}^D \),

\[ \eta (\theta) n_B^I = \frac{e \eta (\theta)}{\delta (1 - \delta) \eta (\theta) n_{1 LD}^I} = L_{BE}^S (\theta; n_{1 LD}^D), \]

\[ \eta (\theta) n_B^L = \mu (\theta) \frac{n_{1 LD}^D n_{1 LD}}{n_B^D + n_{1 LD}} = R_{BE}^S (\theta; n_{1 LD}^D). \]
\[ \eta(\theta)n^I_B = \mu(\theta) \frac{(n^D)^2}{n^D + n^I_1^{LD}} = R^S_{LB}(\theta; n^I_1^{LD}). \]

Set \( n^I_1^{LD} \) equal to the equilibrium value. Equilibrium \( \theta \) satisfies \( L^S_{LB}(\theta; n^I_1^{LD}) = R^S_{LB}(\theta; n^I_1^{LD}) \).

Note that \( L^S_{LB}(\theta; n^I_1^{LD}) \) is increasing and \( R^S_{LB}(\theta; n^I_1^{LD}) \) is decreasing in \( \theta \), and that both \( L^S_{SB}(\theta; n^I_1^{LD}) \) and \( R^S_{SB}(\theta; n^I_1^{LD}) \) are decreasing in \( A \), where \( n^I_1^{LD} \) is increasing in \( A \). This implies that \( \eta(\theta)n^I_B \) is decreasing in \( A \).

**Balanced Eq.** In the Balanced Eq., \( n^I_0^{LD} = n^I_2^{LD} = 0 \) and \( n^I_1^{LD} = n^{LD} \) so \( n^D_B = n^{LD} + n^0_D \) and \( n^D_S = n^{LD} + n^1_SD \). In this case, by means of (32) and (34),

\[
TV = \left\{ \begin{array}{ll}
\eta(n) n^I_1^{LD} & \leq \frac{n^2_D}{2} \\
\eta(n) n^I_2^{LD} & \geq \frac{n^2_D}{2} 
\end{array} \right.
\]

whereas (113) and (114) specialize to, respectively,

\[ \eta(\theta)n^I_B = \frac{\epsilon\eta(\theta)}{\delta + (1 - \delta)\eta(\theta)n^I_1^{SD} + n^D} \equiv L^BA_{LB}(\theta; n^1_SD), \]

\[ \eta(\theta)n^I_S = \frac{\epsilon\eta(\theta)}{\delta + (1 - \delta)\eta(\theta)n^I_1^{SD} + n^D} \equiv R^BA_{LB}(\theta; n^1_SD). \]

Set \( n^1_SD \) equal to the equilibrium value. Equilibrium \( \theta \) satisfies \( L^BA_{LB}(\theta; n^1_SD) = R^BA_{LB}(\theta; n^1_SD) \).

Note that \( L^BA_{LB}(\theta; n^1_SD) \) is increasing and \( R^BA_{LB}(\theta; n^1_SD) \) is decreasing in \( \theta \). As \( A \) increases, both curves shift downward since in the Balanced Eq. \( n^1_SD \) is increasing in \( A \) as implied by (78) or (81). This implies that \( \eta(\theta)n^I_B \) and \( TV \) for \( n^1_SD \leq \frac{n^2_D}{2} \) are decreasing in \( A \).

On the other hand, from (111) and (112), respectively, where \( n^D_B = n^{LD} + n^0_SD \) and \( n^D_S = n^{LD} + n^1_SD \),

\[ \eta(\theta)n^I_S = \frac{\epsilon\eta(\theta)n^I_1^{SD} + n^I_1^{LD}}{\delta + (1 - \delta)\eta(\theta)n^I_1^{SD} + n^D} \equiv L^BA_{LS}(\theta; n^1_SD), \]

\[ \eta(\theta)n^I_S = \frac{\epsilon\eta(\theta)n^I_1^{SD} + n^I_1^{LD}}{\delta + (1 - \delta)\eta(\theta)n^I_1^{SD} + n^D} \equiv R^BA_{LS}(\theta; n^1_SD). \]

Set \( n^1_SD \) equal to the equilibrium value. Equilibrium \( \theta \) satisfies \( L^BA_{LS}(\theta; n^1_SD) = R^BA_{LS}(\theta; n^1_SD) \).

Note that \( L^BA_{LS}(\theta; n^1_SD) \) is increasing and \( R^BA_{LS}(\theta; n^1_SD) \) is decreasing in \( \theta \). As \( A \) increases, both curves shift upward. This implies that \( \eta(\theta)n^I_S \) and \( TV \) for \( n^1_SD \geq \frac{n^2_D}{2} \) are increasing in \( A \).
Buying Eq. In the Buying Eq., \( n_0^{SD} = n_0^{LD} = 0 \) and \( n_1^{SD} = n^{SD} \) and so \( n_B^D = n_1^{LD} \) and \( n_S^D = n^D \). In this case, by means of (26) and (32),

\[
TV = m_0^{SD} + m_0^{LD} = \eta(\theta) \frac{n_B^I}{n^I} (n^{SD} + n_1^{LD}),
\]

whereas (111) and (112) specialize to, respectively,

\[
\eta(\theta)n_B^I = \frac{e \eta(\theta)}{\delta + (1 - \delta) \eta(\theta)} \equiv L_B^{BuE}(\theta),
\]

\[
\eta(\theta)n_S^I = \mu(\theta) \frac{n_D n_1^{LD}}{n^D + n_1^{LD}} \equiv R_B^{BuE}(\theta; n_1^{LD}).
\]

Set \( n_1^{LD} \) equal to the equilibrium value. Equilibrium \( \theta \) satisfies \( L_B^{BuE}(\theta) = R_B^{BuE}(\theta; n_1^{LD}) \).

Note that \( L_B^{BuE}(\theta) \) is increasing and \( R_B^{BuE}(\theta; n_1^{LD}) \) is decreasing in \( \theta \). As \( A \) increases, \( L_B^{BuE}(\theta) \) remains unchanged while \( R_B^{BuE}(\theta; n_1^{LD}) \) goes down because \( n_1^{LD} \) is decreasing in \( A \) in the Buying Eq. as implied by (87). This implies that \( TV \) and \( \eta(\theta)n_B^I \) are decreasing in \( A \).

On the other hand, from (113) and (114), respectively, where \( n_B^D = n_1^{LD} \) and \( n_S^D = n^D \),

\[
\eta(\theta)n_S^I = \frac{e \eta(\theta) n_D n_1^{LD}}{n_D + n_1^{LD}} \equiv L_S^{BuE}(\theta; n_1^{LD}),
\]

\[
\eta(\theta)n_S^I = \mu(\theta) \frac{(n_D)^2}{n^D + n_1^{LD}} \equiv R_S^{BuE}(\theta; n_1^{LD}).
\]

Set \( n_1^{LD} \) equal to the equilibrium value. Equilibrium \( \theta \) satisfies \( L_S^{BuE}(\theta; n_1^{LD}) = R_S^{BuE}(\theta; n_1^{LD}) \).

Note that \( L_S^{BuE}(\theta; n_1^{LD}) \) is increasing and \( R_S^{BuE}(\theta; n_1^{LD}) \) is decreasing in \( \theta \). As \( A \) increases, both \( L_S^{BuE}(\theta; n_1^{LD}) \) and \( R_S^{BuE}(\theta; n_1^{LD}) \) go down. This implies that \( \eta(\theta)n_S^I \) is increasing in \( A \).

This completes the proof of Proposition 7.

**Proof of Corollary 3** When \( A \) changes between 0 and \( A_S \), the Selling Equilibrium holds, in which \( n_0^{SD} = n^{SD} \) and \( n_0^{LD} + n_1^{LD} = n^{LD} \). We need to show that \( n_1^{LD} \) is increasing in \( A \). This follows from (72) because the LHS is decreasing in \( n_1^{LD} \) and increasing in \( A \) while the RHS is increasing in \( n_1^{LD} \) and decreasing in \( A \) over the relevant range of parameters.

When \( A \) changes between \( A_S \) and \( A_B \), the Balanced Equilibrium holds, in which \( n_1^{LD} = n^{LD} \) and \( n_0^{SD} + n_1^{SD} = n^{SD} \). We need to show that \( n_1^{SD} \) is increasing in \( A \). When \( A < \frac{\xi + n^{LD} + n_1^{SD}}{2} \), the LHS of (78) is decreasing in \( n_1^{SD} \) and increasing in \( A \) while the RHS is increasing in \( n_1^{SD} \) and...
decreasing in \( A \) over the relevant range of parameters. When \( A > \frac{\xi}{\delta} + n^{LD} + \frac{n^{SD}}{2} \), the LHS of (81) is increasing in \( n_1^{SD} \) and decreasing in \( A \) while the RHS is decreasing in \( n_1^{SD} \) and increasing in \( A \) over the relevant range of parameters. Finally, \( n_1^{SD} \) is continuous at \( A = \frac{\xi}{\delta} + n^{LD} + \frac{n^{SD}}{2} \).

The above together imply that \( n_1^{SD} \) is increasing in \( A \).

When \( A \) rises above \( A_B \), the Buying Equilibrium holds, in which \( n_1^{SD} = n^{SD} \) and \( n_1^{LD} + n_2^{LD} = n^{LD} \). We need to show that \( n_1^{LD} \) is decreasing in \( A \). This follows from (87), because the LHS is decreasing in both \( n_1^{LD} \) and \( A \) and the RHS is increasing in both \( n_1^{LD} \) and \( A \) over the relevant range of parameters.

**Proof of Proposition 8**  The controls of the planning problem (29) are \( \{ n_0^{SD} (t), n_1^{SD} (t), n_0^{LD} (t), n_1^{LD} (t), n_2^{LD} (t) \} \) and the state variables are \( \{ n_H^{ON} (t), n_S^I (t), n_B^I (t) \} \), where the equations of motion are given by,

\[
\begin{align*}
  n_H^{ON} (t + 1) - n_H^{ON} (t) &= -\delta n_H^{ON} (t) + n_B^I (t) \eta (\theta (t)) \frac{n_S^D (t)}{n_D^D}, \\
  n_S^I (t + 1) - n_S^I (t) &= \delta n_H^{ON} (t) - n_S^I (t) \eta (\theta (t)) \frac{n_B^D (t)}{n_D^D}, \\
  n_B^I (t + 1) - n_B^I (t) &= e - \left( \delta + \eta (\theta (t)) \frac{n_S^D (t)}{n_D^D} \right) n_B^I (t).
\end{align*}
\]

Given the definitions of \( \theta (t) \) in (1), the adding up constraints in (20)-(22) can be summarized by the following two equations:

\[
\begin{align*}
  \eta (\theta (t)) \frac{n_S^D (t)}{n_D^D} &= \eta \left( \frac{n_D^D}{n_S^I (t) + n_B^I (t)} \right) \frac{n_D^D - n_0^{SD} (t) - n_0^{LD} (t)}{n_D^D}, \\
  \eta (\theta (t)) \frac{n_B^D (t)}{n_D^D} &= \eta \left( \frac{n_D^D}{n_S^I (t) + n_B^I (t)} \right) \frac{n_D^D + n^{LD} - A - n_0^{LD} (t) + n_S^I (t) + n_H^{ON} (t)}{n_D^D}.
\end{align*}
\]

Write \( N_S^D (t) \) for \( \eta (\theta (t)) \frac{n_S^D (t)}{n_D^D} \) and \( N_B^D (t) \) for \( \eta (\theta (t)) \frac{n_B^D (t)}{n_D^D} \). In the above, a pair of \( \{ n_0^{SD} (t), n_0^{LD} (t) \} \) uniquely determines the pair \( \{ N_S^D (t), N_B^D (t) \} \). This means that the controls can be stated in terms of \( \{ N_S^D (t), N_B^D (t) \} \), whereby the admissible values are given by

\[
\begin{align*}
  N_S^D (t) &\in \left[ \eta \left( \frac{n_D^D}{n_S^I (t) + n_B^I (t)} \right) \frac{n_D^D (n_H^{ON} (t), n_S^I (t))}{n_D^D}, \eta \left( \frac{n_D^D}{n_S^I (t) + n_B^I (t)} \right) \frac{n_D^D (n_H^{ON} (t), n_S^I (t))}{n_D^D} \right], \\
  N_B^D (t) &\in \left[ \eta \left( \frac{n_D^D}{n_S^I (t) + n_B^I (t)} \right) \frac{n_D^D (n_H^{ON} (t), n_S^I (t))}{n_D^D}, \eta \left( \frac{n_D^D}{n_S^I (t) + n_B^I (t)} \right) \frac{n_D^D (n_H^{ON} (t), n_S^I (t))}{n_D^D} \right],
\end{align*}
\]

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with $\bar{\pi}_S^D$ and $\underline{\pi}_S^D$ denoting, respectively, the largest and smallest possible $n_S^D$ and $\bar{\pi}_B^D$ and $\underline{\pi}_B^D$ denoting, respectively, the largest and smallest $n_B^D$, given state variables $n_H^{ON}(t)$ and $n_S^I(t)$. Let $A^D(t) = A - n_H^{ON}(t) - n_S^I(t)$ be the inventory of asset to be held by dealers. Note that:

1. To attain $\bar{\pi}_S^D$, first allocate one unit each of $A^D(t)$ to either small or large dealers, and then allocate one more unit each to large dealers if $A^D(t) > n^D$.

2. To attain $\underline{\pi}_S^D$, first allocate two units each of $A^D(t)$ to large dealers, and then allocate one unit each to small dealers if $A^D(t) > 2n^{LD}$.

3. To attain $\bar{\pi}_B^D$, first allocate one unit each of $A^D(t)$ to large dealers, and then allocate one unit each to either large or small dealers if $A^D(t) > n^{LD}$.

4. To attain $\underline{\pi}_B^D$, first allocate one unit each of $A^D(t)$ to small dealers, and then allocate two units each to large dealers if $A^D(t) > n^{SD}$.

To proceed, write (29) as

$$W(n_H^{ON}(t), n_S^I(t), n_B^I(t)) = \max_{n_B^I(t), n_B^N(t)} \{ n_H^{ON}(t) \nu + \beta W(n_H^{ON}(t+1), n_S^I(t+1), n_B^I(t+1)) \},$$

in which the state variables for $t + 1$ can be recovered from the equations of motions. There are four constraints corresponding to the four bounds of $n_S^I(t)$ and $n_B^I(t)$. Let $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and $\lambda_4(t)$ be the respective Lagrange multipliers of the lower and upper bounds of $n_S^I(t)$ ($n_B^I(t)$ and $\bar{n}_B^I(t)$) and the lower and upper bounds of $n_B^I(t)$ ($\underline{n}_B^I(t)$ and $\bar{n}_B^I(t)$).

Restricting attention to the steady state, we omit all time indices in the following. The first order conditions for $n_S^D(t)$ and $n_B^D(t)$ are then given by, respectively,

$$\beta n_B^I(W_1 - W_3) + \lambda_1 - \lambda_2 = 0, \tag{122}$$
$$-\beta n_S^I W_2 + \lambda_3 - \lambda_4 = 0. \tag{123}$$

In addition, there are three envelope conditions, one for each state variable:

$$W_1 = \nu + \beta(1 - \delta)W_1 + \beta \delta W_2 - \lambda_1 \frac{\partial n_D^D}{\partial n_H^{ON}} + \lambda_2 \frac{\partial n_D^D}{\partial n_H^{ON}} - \lambda_3 \frac{\partial n_B^D}{\partial n_H^{ON}} + \lambda_4 \frac{\partial n_B^D}{\partial n_H^{ON}}, \tag{124}$$
$$W_2 = \beta(1 - n_B^D)W_2 - \lambda_1 \frac{\partial n_B^D}{\partial n_S^I} + \lambda_2 \frac{\partial n_B^D}{\partial n_S^I} - \lambda_3 \frac{\partial n_B^D}{\partial n_S^I} + \lambda_4 \frac{\partial n_B^D}{\partial n_S^I}, \tag{125}$$
$$W_3 = \beta n_S^D W_1 + \beta (1 - \delta - n_S^D) W_3 - \lambda_1 \frac{\partial n_S^D}{\partial n_B^I} + \lambda_2 \frac{\partial n_S^D}{\partial n_B^I} - \lambda_3 \frac{\partial n_S^D}{\partial n_B^I} + \lambda_4 \frac{\partial n_S^D}{\partial n_B^I}. \tag{126}$$
We first show that $\lambda_3$ must equal to 0. Suppose otherwise. Then, by the definitions of $\pi_{S}^D$, $n_{S}^D$, $\pi_{B}^D$ and $n_{B}^D$, $\lambda_1$, $\lambda_2$ and $\lambda_4$ must all equal to 0 and so equations (122)-(126) reduce to four equations, (123)-(125) with only three unknowns, $\lambda_3$, $W_1$ and $W_2$. The set of $\lambda_3$, $W_1$ and $W_2$ that satisfy all three equations is of measure zero. Therefore, $\lambda_3$ must equal to 0.

Next, we show that $\lambda_1$ must equal to 0. Suppose otherwise. Then, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and so (123) says that $W_2 = 0$, which further implies that $\lambda_1 = 0$ from (125). This is a contradiction and so $\lambda_1 = 0$.

Next, we show that $\lambda_2 > 0$. Suppose otherwise. Together with the fact that $\lambda_1 = 0$, (122) becomes $W_1 = W_3$. We already know that $\lambda_3 = 0$. Then, (122)-(126) reduce to four equations (123)-(126) in three unknowns, $\lambda_4$, $W_1$ and $W_2$. We reach the desired contradiction.

Finally, we show that $\lambda_4 > 0$. Suppose otherwise. Then, by equation (123), $W_2 = 0$. Substituting it into equation (125) implies that $\lambda_2 = 0$, which contradicts our previous conclusion that $\lambda_2 > 0$.

To summarize, we have shown that $N_S^D (t) = \eta \left( \frac{n_{S}^D}{n_{S}^D(t) + n_{B}^D(t)} \right) \frac{\pi_{B}^D(n_{B}^N(t), n_{S}^I(t))}{n_{D}}$ and $N_B^D (t) = \eta \left( \frac{n_{B}^D}{n_{S}^D(t) + n_{B}^D(t)} \right) \frac{\pi_{B}^D(n_{B}^N(t), n_{S}^I(t))}{n_{D}}$. In other words, for efficiency, we should allocate the assets held by dealers to maximize the measure of dealer-sellers and dealer-buyers: first allocate one unit each to large dealers; if $A^D > n_{LD}$, then allocate one unit each to small dealers; if $A^D > n_{D}$, then allocate one more unit each to large dealers. The allocation is the same as the allocation as described in the discussions following Proposition 3.

Proof of Proposition 9  We first verify that any investor-dealer match between a seller and a buyer yields a non-negative surplus in any equilibrium in which both small and large dealers are active agents, selling to and buying from investors. To begin, the surpluses of the possible trades are as follows.

\[
\begin{align*}
z_{I_B,S_1} & = U_H - U_B - (V_1^{SD} - V_0^{SD}), \\
z_{I_B,L_i} & = U_H - U_B - (V_i^{LD} - V_{i-1}^{LD}) \quad \text{for} \quad i = 1, 2, \\
z_{S_0,L_i} & = V_i^{SD} - V_0^{SD} - U_S, \\
z_{L_i,S_0} & = V_{i+1}^{LD} - V_i^{LD} - U_L \quad \text{for} \quad i = 0, 1, \\
z_{S_0,L_i} & = V_1^{SD} - V_0^{SD} - (V_i^{LD} - V_{i-1}^{LD}) \quad \text{for} \quad i = 1, 2,
\end{align*}
\]
\[ z_{L_i,S_1} = V_{i+1}^{LD} - V_i^{LD} - (V_i^{SD} - V_0^{SD}) \text{ for } i = 0, 1, \]
\[ z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - (V_2^{LD} - V_1^{LD}), \]
\[ z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - (V_1^{LD} - V_0^{LD}). \]

An \( S_1 \) can sell to an \( I_B \), an \( L_0 \), or an \( L_1 \). He will not sell to an \( I_B \) only if selling to other dealers yields a strictly larger surplus; i.e.,

\[ \max \{ z_{L_0,S_1}, z_{L_1,S_1} \} > z_{I_B,S_1}. \]

Expanding the expressions for the zs,

\[ \max \{ V_1^{LD} - V_0^{LD}, V_2^{LD} - V_1^{LD} \} > U_H^{ON} - U_B^I. \]

Subtracting \( U_H^{ON} - U_B^I \) from the two sides of the condition,

\[ \max \{ V_1^{LD} - V_0^{LD} - (U_H^{ON} - U_B^I), V_2^{LD} - V_1^{LD} - (U_H^{ON} - U_B^I) \} > 0. \]

The two terms inside the max operator are simply the negatives of \( z_{I_B,L_1} \) and \( z_{I_B,L_2} \), respectively. Then, the condition becomes

\[ \max \{-z_{I_B,L_1}, -z_{I_B,L_2}\} > 0 \Leftrightarrow \min \{ z_{I_B,L_1}, z_{I_B,L_2} \} < 0. \]

All this implies that if one type of dealer-seller finds it optimal not to sell to investor-buyers, then only one type of dealer-seller may find it optimal to do so. In any steady-state equilibrium that involves trading between dealers and investors, indeed at least one type of dealer-seller must do so.

Now, suppose only \( S_1 \)s sell to \( I_B \) where

\[ z_{I_B,S_1} = U_H^{ON} - U_B^I - (V_1^{SD} - V_0^{SD}) \geq 0. \tag{127} \]

An \( L_1 \) may then only sell to an \( S_0 \) or another \( L_1 \). Selling to an \( S_0 \) is optimal if

\[ z_{S_0,L_1} = V_1^{SD} - V_0^{SD} - (V_1^{LD} - V_0^{LD}) \geq 0. \]

But if the condition holds,

\[ z_{I_B,L_1} = U_H^{ON} - U_B^I - (V_1^{LD} - V_0^{LD}) \geq 0 \]

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must hold given (127). The hypothesis that only $S_1$ sell to $I_B$s then implies that selling to another $L_1$ must be optimal for the $L_1$ (otherwise the $L_1$ has no one to sell to), where

$$z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - (V_1^{LD} - V_0^{LD}) \geq 0. \quad (128)$$

An $L_2$ may sell to an $S_0$ or an $L_0$ if selling to an $I_B$ is not optimal. Selling to an $S_0$ is optimal if

$$z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - (V_2^{LD} - V_1^{LD}) \geq 0.$$

But if the condition holds,

$$z_{I_B,L_2} = U_H^{ON} - U_B^I - (V_2^{LD} - V_1^{LD}) \geq 0$$

must hold given (127). The hypothesis that only $S_1$s sell to $I_B$s then implies that selling to an $L_0$ must be optimal for the $L_2$, where

$$z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - (V_2^{LD} - V_1^{LD}) \geq 0. \quad (129)$$

The two conditions, (128) and (129), together imply that

$$V_1^{LD} - V_0^{LD} = V_2^{LD} - V_1^{LD}.$$ 

Thus, if neither $L_1$s nor $L_2$s find it optimal to sell to investor-buyers or to small dealers, large dealers do not gain by selling and buying among themselves either. They must then be inactive in equilibrium.

Next, suppose only $L_1$s sell to investor-buyers, where

$$z_{I_B,L_1} = U_H^{ON} - U_B^I - (V_1^{LD} - V_0^{LD}) \geq 0. \quad (130)$$

An $S_1$ may sell to an $L_0$ or to an $L_1$ if not selling to an investor-buyer. If the first sale is optimal, it must be optimal for the $S_1$ to sell to an $I_B$ as well given (130). The hypothesis that only $L_1$s sell to investor-buyers then requires that it is optimal for an $S_1$ to sell to an $L_1$ where

$$z_{L_1,S_1} = V_2^{LD} - V_1^{LD} - (V_1^{SD} - V_0^{SD}) \geq 0. \quad (131)$$
An $L_2$ may sell to an $L_0$ or to an $S_0$. If the first sale is optimal, it must be optimal for the $L_2$ to sell to an $I_B$ as well given (130). The condition for the second sale to be optimal is that

$$z_{S_0,L_2} = V_{1SD} - V_{0SD} - (V_{2LD} - V_{1LD}) \geq 0. \quad (132)$$

The two conditions, (131) and (132), together imply that

$$V_{1SD} - V_{0SD} = V_{2LD} - V_{1LD}.$$

Thus, if neither $S_1$s nor $L_2$s find it optimal to sell to investor-buyers, $S_1$s only sell to $L_1$s, where such trades do not yield any surplus. This implies that small dealers must be inactive in equilibrium.

The case for where only $L_2$s sell to investor-buyers can be shown in a similar way to imply that small dealers must be inactive in equilibrium.

The proof that in any equilibrium in which both small and large dealers are active, investor-sellers must sell to all three types of dealer-buyers can be constructed similarly.

Given that all investor-dealer trades shall yield a non-negative match surplus, with Nash Bargaining and each agent in a match entitled to one-half of the match’s surplus, we can rewrite dealers’ value functions as follows.

$$rV_{0SD} = \eta(\theta) \frac{n^I_S}{n^D} z_{S_0,I_S} \left( \frac{2}{n^D} \right) + \alpha \left\{ \frac{n_{1LD}}{2n^D} \max\{z_{S_0,L_1}, 0\} + \frac{n_{1LD}}{2n^D} \max\{z_{S_0,L_2}, 0\} \right\},$$

$$rV_{1SD} = \eta(\theta) \frac{n^I_B}{n^D} z_{I_B,S_1} \left( \frac{2}{n^D} \right) + \alpha \left\{ \frac{n_0^{LD}}{2n^D} \max\{z_{L_0,S_1}, 0\} + \frac{n_{1LD}}{2n^D} \max\{z_{L_1,S_1}, 0\} \right\},$$

$$rV_{0LD} = \eta(\theta) \frac{n^I_S}{n^D} z_{L_0,I_S} \left( \frac{2}{n^D} \right) + \alpha \left\{ \frac{n_1^{SD}}{2n^D} \max\{z_{L_0,S_1}, 0\} + \frac{n_{2LD}}{2n^D} \max\{z_{L_0,L_2}, 0\} \right\},$$

$$rV_{1LD} = \eta(\theta) \frac{n^I_B}{n^D} z_{I_B,L_1} \left( \frac{2}{n^D} \right) \left( \frac{2}{n^D} \right) + \alpha \left\{ \frac{n_0^{SD}}{2n^D} \max\{z_{S_0,L_1}, 0\} + \frac{n_1^{SD}}{2n^D} \max\{z_{L_1,S_1}, 0\} + \frac{n_{1LD}}{2n^D} \max\{z_{L_1,L_1}, 0\} \right\},$$

$$rV_{2LD} = \eta(\theta) \frac{n^I_B}{n^D} z_{I_B,L_2} \left( \frac{2}{n^D} \right) \left( \frac{2}{n^D} \right) + \alpha \left\{ \frac{n_0^{SD}}{2n^D} \max\{z_{S_0,L_2}, 0\} + \frac{n_1^{SD}}{2n^D} \max\{z_{L_0,S_2}, 0\} + \frac{n_{2LD}}{2n^D} \max\{z_{L_0,L_2}, 0\} \right\}.$$

Suppose $V_{1SD} - V_{0SD} > V_{1LD} - V_{0LD}$. Then $z_{I_B,S_1} < z_{I_B,L_1}$ and $z_{S_0,I_S} > z_{L_0,I_S}$. Together with the fact that $z_{L_1,I_S} \geq 0$, this implies that

$$\eta(\theta) \frac{n^I_B}{n^D} z_{I_B,S_1} \left( \frac{2}{n^D} \right) - \eta(\theta) \frac{n^I_S}{n^D} z_{S_0,I_S} \left( \frac{2}{n^D} \right) < \eta(\theta) \frac{n^I_S}{n^D} z_{L_1,I_S} \left( \frac{2}{n^D} \right) + \eta(\theta) \frac{n^I_B}{n^D} z_{I_B,L_1} \left( \frac{2}{n^D} \right) - \eta(\theta) \frac{n^I_S}{n^D} z_{L_0,I_S} \left( \frac{2}{n^D} \right).$$

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Also, \( V_1^{SD} - V_0^{SD} > V_1^{LD} - V_0^{LD} \) implies that \( z_{S_0,L_1} > 0 > z_{L_0,S_1}, z_{S_0,L_2} > z_{L_0,L_2} \), and \( z_{L_1,S_1} < z_{L_1,L_1} \). This means

\[
\frac{n_1^{LD}}{2n_D} \max\{z_{S_0,L_1}, 0\} + \frac{n_2^{LD}}{2n_D} \max\{z_{S_0,L_2}, 0\} > \frac{n_1^{SD}}{2n_D} \max\{z_{L_0,S_1}, 0\} + \frac{n_2^{SD}}{2n_D} \max\{z_{L_0,L_2}, 0\}
\]

and

\[
\frac{n_0^{LD}}{2n_D} \max\{z_{L_0,S_1}, 0\} + \frac{n_1^{LD}}{2n_D} \max\{z_{L_1,S_1}, 0\} < \frac{n_0^{SD}}{2n_D} \max\{z_{S_0,L_1}, 0\} + \frac{n_1^{SD}}{2n_D} \max\{z_{S_1,L_1}, 0\} + \frac{n_2^{LD}}{2n_D} \max\{z_{L_1,L_1}, 0\}
\]

The above three inequalities together imply that \( V_1^{SD} - V_0^{SD} < V_1^{LD} - V_0^{LD} \). This is a contradiction.

Now suppose \( V_2^{LD} - V_1^{LD} > V_1^{SD} - V_0^{SD} \). Similarly, we can show that this implies \( z_{L_1,I_1} > z_{S_0,I_1} > z_{I_1,L_1} \), \( z_{S_0,I_1} < z_{I_1,S_1}, z_{S_0,L_2} < 0 < z_{L_0,S_1} \) and \( z_{L_0,L_2} < z_{L_0,S_1} \). These inequalities in turn imply that \( V_2^{LD} - V_1^{LD} < V_1^{SD} - V_0^{SD} \). This is a contradiction.

Given that we have shown \( V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD} \), it is straightforward to verify that the two equalities hold are strict unless \( z_{I_1,L_1} = 0 \).

References


