Thick Market Externalities in a Spatial Model

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Abstract

It is natural to think of thick market externalities as spatial phenomena. When agents are in close physical proximity, potential trading partners are more numerous and less costly to reach. Counteracting such agglomeration benefits is the dispersion force due to land being an essential input in production. The distribution of economic activities over space is an outcome of how decisions on location, land demand, and the search strategy of agents interact in spatial equilibrium. More desirable locations are those that allow their occupants more abundant and less costly access to potential trading partners. In spatial equilibrium, these are the densest locations, the occupants of which benefit from the strongest thick market externalities.

Keywords: agglomeration economies, search and matching, density, thick market externalities

JEL classification: R12, R13, R14

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1. INTRODUCTION

It is natural to think of thick market externalities, those forces that give rise to more opportunities for exchange in a market with an increase in the size of the market, as spatial phenomena. Interactions are most frequent and take place at the lowest cost when people cluster. The logical conclusion of such an argument, of course, is that, other things being equal, it is best for people to all stay at a single spot as workplace to maximize trading opportunities and minimize the cost of trade. Obviously, such complete concentration of economic activities is precluded by land being an input in production. Thus, whereas concentration promotes interaction and helps strengthen any thick market externalities that may be in place, dispersion relieves crowding and can lower the cost of production. A basic force that determines the strength of thick market externalities then is how people decide where to stay and how much land to occupy.

This paper studies a model of how decisions on location, land demand, and the search strategy of economic agents interact to determine the density of economic activities across space, on which trading opportunities and the cost of trade for agents at various locations in a regional economy depend. The analysis is based on the model of production and exchange through search and matching in Diamond (1982), in which an agent carrying a unit inventory of output must seek another agent for exchange and consumption before she may begin production of the next unit of output. To the Diamond model I add the assumption that the production and exchange take place in a spatial economy, where each agent occupies a positive amount of land while searching for a trading partner – an assumption made to model the fact that land is often an essential input in production. In the formal analysis, I assume that land inputs are used for the maintenance of inventories rather than physical production. The calculations to follow are simpler, while the conclusions should apply equally well to the case where land is indeed used for production.

The thick market externality in the model comes from the assumptions that: (i) the rate at which an agent is matched with a potential trading partner increases with an increase in the population mass in the area across which the given agent searches; (ii) the cost to complete a bilateral trade is proportional to the distance separating the two agents in the trade. Two forces follow. First, agents will seek to move to locations around which potential trading partners are most numerous and least costly to reach. Second, it becomes important for agents to choose the areas across which the search should be carried out. To minimize the cost of trade, an agent may choose to search for a trading partner only in a small area centered around her own location. However, searching over a larger area can help one to conclude the search sooner as more people are included as potential trading partners. In sum, in a model economy in which search and spatial frictions interact, individual agents must make key decisions about: (1) location, (2) land demand, and (3) the distance that the search should cover.

The analysis is simplest when everyone chooses to search from one to the other end of a region. In that case, the matching rate should be the same for everyone. However the cost of trade may not be the same. As long as the location distribution of agents is symmetric around the regional center, it is least costly to trade on average when one is right at the regional center. In the ensuring competition in the land market, there will be a downward-sloping rent gradient centered at the midpoint of the region. The higher land rents in turn will induce the occupants of central locations to substitute away from land for non-land inputs. There will then be a unimodal distribution of agents across space, whereby density decreases everywhere with increasing distance from the regional center.

With an exogenous increase in the regional population, everyone benefits by being able to trade more frequently. Meanwhile, the region expands horizontally. At some point in the expansion, agents at certain locations will no longer find it worthwhile to search over all locations in the larger region. If not all agents search over the same set of locations, a priori it is no longer obvious that central locations will offer their occupants less costly access to trading partners. Then it is not clear that a unimodal density centered at the midpoint of the region is an inevitable outcome of spatial equilibrium. Moreover, should the density turn out to be of any other shape, it is not even clear that central locations can offer the most abundant trading opportunities for their occupants.

A subtle but nevertheless intuitive result of the analysis is that central locations do turn out to be the best locations in spatial equilibrium even if agents at various locations endogenously choose to interact only with others in their vicinity. In a spatial-trade model, when one location becomes more attractive, so do neighboring locations. Such interdependence suffices to lead to a unimodal density, which decreases everywhere with increasing distance from the regional center.

If agents at denser locations experience stronger thick market externalities and thus are able to trade more frequently, then their rates of output rise concomitantly. Hence, a spatial model of thick market externalities is also a model of agglomeration economies, with a positive relation between population and density on the one hand and productivity on the other hand. Although the link between agglomeration and productivity in the present model hinges on the assumption of the Diamond model that an agent may resume production only after her last unit of output is successfully "sold", this assumption is not at all unrealistic and can easily be justified by a certain cash-flow constraint that prevents firms from accumulating inventories indefinitely.

The mechanism of agglomeration economies explored in this paper is a variant of the matching mechanism of agglomeration economies in Kim (1989, 1991), Helsley and Strange (1990), and Berliant et al. (2006), where clustering helps raise productivity by improving the quality of matches. In the present model, clustering helps raise productivity by enabling producers to match more frequently.

What sets the present model apart from previous models of agglomeration economies is that in the former, space and density are basic elements underlying the local increasing returns. This is in contrast to the usual modeling practice in this strand of investigation, in which space and density do not feature explicitly in the mechanism of the local increasing returns but appear only in the mechanism of the centrifugal force that restricts the size of the urban center. For instance, in the typical model, firms are assumed to all cluster around a dimensionless city center, and that it is the overall scale of economic activities in the given city that determines the strength of the local increasing returns. The given production structure is then embedded in an urban model in which workers commute to the city center for employment.[1] In the present model, each economic agent takes up a positive amount of space as a producer. It is the extent to which production units cluster that determines the strength of the local increasing returns as well as the severity of the urban congestion that may restrict the extent of agglomeration.

The formal structure of the model is similar to that of models of the interaction of agents over space in Solow and Vickrey (1971), Beckmann (1976), and Borukhov and Hochman (1977). Assuming that everyone travels to interact with all others in a given locale, these models study the equilibrium location distribution of households and firms. More recently, Helsley and Strange (2007) extend such analysis by allowing households to choose the frequency of visits made to the city center. In the present paper, agents choose how far away from their own locations that they will travel, and thus may choose to interact only with a subset of all agents in the given locale. This is related to but distinct from Coulson et al. (2001), who study how workers choose

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[1] Important exceptions include Ogawa and Fujita (1980), Fujita and Ogawa (1982), Lucas (2000), and Berliant et al. (2002), among others. However, the distance-dependent production externalities in these models are assumed rather than derived.

between searching for jobs in either one of two given employment centers in a city.

By modeling how searches take place over a spatial economy, this paper is related to the urban labor market literature, which includes, among others, Wasmer and Zenou (2002), who study whether more central locations are occupied by employed or unemployed workers, and Brueckner et al. (2002) and Zenou (2009), who find that more productive workers reside closer to the employment center than do less productive ones. Similar to these models, the present model is concerned with how frictions in non-land markets interact with competition in the land market. In contrast to these models, the present paper does not assume that all economic interactions take place in a single location. More closely related to the present paper are Rouwendal (1998) and Gautier and Zenou (2010), in which workers choose the maximum commute to tolerate in a labor market with search frictions in much the same way that agents in the present model choose the maximum distance over which to conduct their search. In accepting a job offer, workers in the models of Rouwendal (1998) and Gautier and Zenou (2010) are committing to a long-term relationship while giving up the option to continue searching. Hence, the decision problems are somewhat more complex than those faced by agents in the present model, in which matches are formed and then dissolved in the very next instant. However, locations in these models are exogenous, whereas the present model analyzes the equilibrium location distribution of agents with endogenous space demand. In another strand of investigation, Gautier et al. (2010) and Gautier and Teulings (2009) find that people are more selective in accepting job offers and choosing potential mates for marriage in big cities, where matching opportunities are more abundant. These papers introduce an important manifestation of thick market externalities – how the loss due to less-than-perfect assortative matching may be reduced in thick markets. In contrast, the present paper is concerned with how the strength of thick market externalities is determined in the first place and how it differs across locations within and between regions. As a model of trade in the product market with spatial frictions, this paper is related to Berliant and Wang (1993) and Berliant and Konishi (2000), who study the endogenous formation of marketplaces. In Tse (2010), I study how the interaction of spatial and search frictions may give rise to a role for middlemen in helping others to trade at lower costs.

The rest of the paper is organized as follows. The next section presents the model and solves for the equilibrium location distribution of agents. In section three, I study the relation between productivity and density within and between regions. Section four discusses how equilibrium allocation differs from optimum allocation. Section five looks at alternative assumptions of the spatial structure, search and spatial frictions, and thick market externalities. Section six concludes with some brief remarks. The appendix contains two lemmas on the analytical solutions of the equilibrium location distribution and the proof of Lemma 6 in the main text.

2. MODEL AND SPATIAL EQUILIBRIUM

Production and exchange over space Space is one dimensional. The regional economy, centered at point 0, covers all locations $x \in [-\overline{x}, \overline{x}]$, where $\overline{x}$, a value to be endogenously determined, denotes the distance separating the farthest location in the region from the region's center. A continuum of mass n risk-neutral agents live and work in the regional economy. The agents are ex ante identical, and each discounts the future at the same rate ρ. There is a fixed coefficient technology that enables an agent to produce and then store in inventory one unit of an indivisible special good at a time. People cannot consume their own output and thus must seek others for exchange and consumption.

In particular, when an agent's inventory is empty, she may produce a unit of the special good at a utility cost normalized to zero. Henceforth, the agent must choose a location as the base from which to direct her search for a trading partner and the amount of land input l to employ in maintaining the unit inventory. There are also non-land inputs in inventory management. Let $k(l)$ be the utility cost that the agent incurs per time unit for the deployment of such inputs. Assume that $\partial k(l)/\partial l < 0$. That is, an increase in the employment of land input helps lower the amount of non-land inputs required.

When one's search for a trading partner covers a wider area, potential trading partners are more numerous, which should help one conclude the search sooner. However, often the cost of executing a trade is proportional to the distance separating the two parties in the trade. The downside then to extending the search to more remote locations is that because the eventual trading partner is more likely to be far from one's own locale, the trade is on average more costly to complete.

Specifically, let $H(x)$ be the distribution of agents over space, a twice (piecewise) differentiable and (weakly) increasing function that satisfies[2]

$$H(x) = 0 \text{ for } x \leq -\overline{x}; \quad H(x) = n \text{ for } x \geq \overline{x}. \tag{1}$$

Write the density function as $h(x) = \partial H(x)/\partial x$. Outside the borders of the regional economy, the density is zero; i.e., for $x \notin [-\overline{x}, \overline{x}]$, $h(x) = 0$. If a given agent chooses to search up to a distance of $\overline{\delta}_L$ to the left and a distance of $\overline{\delta}_R$ to the right of where the agent is located, say, at point x, then the set of potential trading partners will comprise all others at locations along the interval $\mathcal{I} \equiv \left[x - \overline{\delta}_L, x + \overline{\delta}_R\right]$. Write

$$m = H\left(x + \overline{\delta}_R\right) - H\left(x - \overline{\delta}_L\right) \tag{2}$$

for the mass of agents residing in the given agent's search area. The distribution of the distance between the given agent and her potential trading partners is given by

$$F(\delta; x) = \frac{H(x + \delta_R) - H(x - \delta_L)}{m}, \tag{3}$$

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[2] More generally, one should write $H(x) = 0$ for $x < -\overline{x}$. The first restriction in (1) is without loss of generality as long as there are no mass points in $H(x)$ – a condition that is guranateed by the assumption that each agent occupies a positive amount of space.

where $\delta_R = \max\left\{\delta, \overline{\delta}_R\right\}$ and $\delta_L = \max\left\{\delta, \overline{\delta}_L\right\}$. Assume that the search within $\mathcal{I}$ is purely random, and that the rate at which the agent successfully contacts someone located within $\mathcal{I}$, denoted as $\beta(m)$, is a strictly increasing function of m.

When one agent successfully contacts another agent, the two may swap their respective unit inventories for consumption. The exchange can only be completed at a cost proportional to the physical distance separating the two agents. Assume for simplicity that the constant of proportionality is just equal to one. How this trade cost is shared between the two agents is a matter of the two bargaining over the surplus of the match. After the trade is completed and the units are consumed, the match dissolves. Each agent replenishes her inventory and chooses the amount of land and non-land inputs and the location from which to carry out the search for the next trading opportunity.[3] With all agents in the regional economy actively seeking out each other, a given agent may trade not only when she successfully contacts another agent but also when she happens to be contacted by someone else in the regional economy. Denote this probability as $\alpha(x)$, which is potentially a function of the agent's location, to the extent that occupants of different locations can be sought at various intensities by others. In this connection, let $G(\delta; x)$ be the distribution of the distance separating the agent at x from the agent who initiates the contact.

All of the land in the regional economy is owned by absentee landlords. Land rents and trade expenses are paid in terms of a numeraire good of which agents in the regional economy can produce any units at a utility cost normalized to the rate of production.

Let $v(x, l)$ be the steady-state value of search for an agent located at x, who has

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[3] One referee pointed out that in many instances, even outside the labor market, the relationship between a pair of matched agents is long lasting, as in a manufacturer-dealer or a bank-investor relationship. The additional complexity arising from modeling long-term matches does not, however, seem to help yield additional insights in the present model.

chosen to employ l units of land in the interim. The choices of location and land input are made to

$$\max_{\{x,l\}} v\left(x,l\right) \equiv w. \tag{4}$$

Let $l\left(x\right) = \arg\max_l v\left(x,l\right)$. Given free mobility, in spatial equilibrium, $v\left(x,l\left[x\right]\right) = w$ for all $x \in \left[-\overline{x},\overline{x}\right]$. Then the match surplus of a meeting between two agents located at x and x', respectively, is given by

$$u - v\left(x,l\left[x\right]\right) + w + u - v\left(x',l\left[x'\right]\right) + w - \left|x - x'\right| = 2u - \delta,$$

where $\delta = \left|x - x'\right|$ is the cost of to complete a trade between the two agents. In the Nash bargaining over the match surplus, each agent will then be liable to pay one half of δ. In all, the steady-state value of $v\left(x,l\left[x\right]\right)$ is given by

$$\begin{aligned} \rho v\left(x,l\left[x\right]\right) &= \max_{\left\{\overline{\delta}_L,\overline{\delta}_R\right\}} \left\{\beta\left(m\right) \times \int_0^\infty \max\left\{u - \frac{\delta}{2},0\right\} dF\left(\delta;x\right)\right. \\ &\qquad + \alpha\left(x\right) \times \int_0^\infty \max\left\{u - \frac{1}{2}\delta,0\right\} dG\left(\delta;x\right) \\ &\qquad \left. - \left(k\left(l\left[x\right]\right) + r\left(x\right) l\left[x\right]\right)\right\}, \end{aligned} \tag{5}$$

where $r\left(x\right)$ is the unit land rent function. The sum of the first two lines of (5) can be considered as the flow payoff of search: at a rate $\beta\left(m\right)$, the agent successfully initiates contact with an agent located within $\mathcal{I}$. If there is a positive surplus, an exchange of inventories takes place at a cost of δ, with the agent at x paying one half of the trade cost. In addition, at a rate $\alpha\left(x\right)$, a rate taken as given by the agent, she is contacted by another agent in the regional economy, located at some distance δ, which is a draw from the distribution $G\left(\delta;x\right)$. Likewise, if there is a positive surplus, a match will be formed and the agent pays one half of the cost to complete the trade. Notice that the agent's decisions are restricted to the choices of β and F, and that she has no control over α and G. Lastly, at each moment of time while searching, the agent incurs a non-land flow expenditure $k\left(l\right)$ and rental expenditure $r\left(x\right) l$ to maintain her unit inventory, the sum of which can be thought of as the cost of search.

Now a more intelligent search strategy than searching randomly along $\mathcal{I}$ would involve the agent first searching among neighboring agents before expanding her search to more distant locations. To model such a search strategy in a tractable manner, one can replace the $F(\delta; x)$ function in (5) by

$$\widetilde{F}(\delta; x) = \frac{\int_0^\delta e^{-\gamma z} dF(z; x)}{\int_0^\infty e^{-\gamma z} dF(z; x)}$$

for some parameter $\gamma > 0$, whereby the agent is more likely to contact a neighboring agent than others at distant locations. The analysis that follows is technical and cumbersome, but the major conclusions are unaltered as long as elements of random search are retained. For brevity, I proceed with the simpler assumption of purely random search underlying (5).

Optimal search In maximizing (5), the agent clearly should constrain the search only up to where $u - \delta/2 \geq 0$, but not any farther. It is not necessarily true, however, that the search should cease at the point at which any match surplus is exhausted. In restricting the search to a smaller area, the saving in the expected trade cost may more than make up for the decrease in the matching rate.

Lemma 1 *If $\beta(m) = \theta m$ for some parameter $\theta > 0$,*

$$\overline{\delta}_L = \overline{\delta}_R = 2u.$$

Proof. By (3),

$$\int_0^\infty \delta dF(\delta; x) = \frac{1}{m}\left(\int_{x-\overline{\delta}_L}^{x} (x-s)\, h(s)\, ds + \int_x^{x+\overline{\delta}_R} (s-x)\, h(s)\, ds\right). \tag{6}$$

By (2),

$$\frac{\partial m}{\partial \overline{\delta}_L} = h\left(x - \overline{\delta}_L\right); \qquad \frac{\partial m}{\partial \overline{\delta}_R} = h\left(x + \overline{\delta}_R\right). \tag{7}$$

Then

$$\frac{\partial\left[\int_0^\infty \delta dF(\delta; x)\right]}{\partial \overline{\delta}_L} = \frac{h\left(x - \overline{\delta}_L\right)}{m}\left(\overline{\delta}_L - \int_0^\infty \delta dF(\delta; x)\right), \tag{8}$$

$$\frac{\partial \left[\int_0^\infty \delta dF\left(\delta;x\right)\right]}{\partial \overline{\delta}_R} = \frac{h\left(x+\overline{\delta}_R\right)}{m}\left(\overline{\delta}_R - \int_0^\infty \delta dF\left(\delta;x\right)\right). \tag{9}$$

Assuming $\beta\left(m\right) = \theta m$, while making use of $(7)-(9)$, the first order conditions of (5) read

$$h\left(x-\overline{\delta}_L\right)\left(2u-\overline{\delta}_L\right) = 0,$$
$$h\left(x+\overline{\delta}_R\right)\left(2u-\overline{\delta}_R\right) = 0,$$

from which the result of the lemma is obtained. ■

Lemma 1 shows that if β is a unitary elastic function of m, an assumption that I maintain hereinafter, then the agent should continue expanding her search to any location at which a trade with its occupant will yield a positive match surplus.[4] In this case,

$$m\left(x\right) = H\left(x+2u\right) - H\left(x-2u\right). \tag{10}$$

Of course, if either one or both of

$$x + 2u > \overline{x}, \tag{11}$$

$$x - 2u < -\overline{x}, \tag{12}$$

hold, then the search effectively terminates at one or both ends of the regional economy, where the land beyond is uninhabited.[5] Notice that the two conditions (11)

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[4] Lemma 1 can be thought of as describing a "corner solution," in which the agent finds it optimal to include anyone with whom a trade would yield a positive match surplus as a potential trading partner. When $\beta\left(m\right)$ only increases less than proportionately with m, at some point, the gains from the increase in the matching rate will not justify the increase in the expected trade cost due to an expansion of the search area.

[5] Although it seems more accurate to write $\overline{\delta}_L = \min\left\{2u, x+\overline{x}\right\}$ and $\overline{\delta}_R = \min\left\{2u, \overline{x}-x\right\}$, where it is futile to search beyond the borders of the regional economy, the simpler expression in Lemma 1 is nevertheless valid. In the present setup, searching over empty territories outside the boundaries of the regional economy is neither wasteful nor useful for both β and $\int \delta dF\left(\delta;x\right)$ remain unaffected.

and (12) are more likely to apply to locations near the right and left borders, respectively, of the regional economy than to central locations. Hence, even when agents are uniformly distributed over space, those at border locations may search only among a smaller set of potential trading partners compared to those at central locations. Such a handicap confronting occupants of border locations is in addition to the usual classical endpoint effect in the location models considered by Beckmann (1976) and Borukhov and Hochman (1977), in which agents at border locations travel longer distances on average to reach others in a linear space.

Optimum land input By (4) and (5), for each x, optimum land input $l(x)$ satisfies

$$-\frac{\partial k\left(l\left[x\right]\right)}{\partial l}=r\left(x\right); \tag{13}$$

i.e., at the optimum, a unit increase in land input gives rise to a decline in non-land expenditure on inventory management just equal to the increase in rental expenditure. At the two borders of the regional economy, the land rents, however, must be equal to the exogenously given rural rent, denoted as $\overline{r}$; i.e.,

$$-\frac{\partial k\left(l\left[-\overline{x}\right]\right)}{\partial l}=-\frac{\partial k\left(l\left[\overline{x}\right]\right)}{\partial l}=\overline{r}. \tag{14}$$

Given a functional form for k, (14) pins down exactly the optimum land inputs for agents right at the borders of the regional economy and hence the border densities $h\left(\overline{x}\right)=1/l\left(\overline{x}\right)=1/l\left(-\overline{x}\right)=h\left(-\overline{x}\right)$. The optimum land inputs for agents at interior locations and hence the equilibrium densities, by (13), are functions of land rents. In turn, in equilibrium, $r\left(x\right)$ must vary to equalize $v\left(x,l\left[x\right]\right)$ for all $x\in\left[-\overline{x},\overline{x}\right]$, a subject to be addressed in the following. But first we need to solve for the equilibrium α and G.

Equilibrium α and G As each agent in the regional economy chooses to search up to a distance of $2u$ from her own location, the set of agents that choose to include someone at x as a potential trading partner comprises all agents residing along $[x-2u, x+2u]$. If an agent located within the interval, say at point s, searches randomly among a mass of $m(s)$ potential trading partners, she will contact a particular individual – someone who is located at x, with probability $1/m(s)$. Thus, the rate at which an agent at x is contacted by someone located within $[x-2u, x+2u]$ is given by

$$\alpha(x) = \int_{x-2u}^{x+2u} \frac{\theta m(s)}{m(s)} dH(s) = \theta [H(x+2u) - H(x-2u)].$$

Notice that the value for α turns out to be just equal to the value for β. This is hardly surprising, given that search areas are symmetric. In this case, clearly, $G(\delta; x) = F(\delta; x)$, too.

Spatial equilibrium With $\alpha(x) = \beta(m[x]) = \theta m(x)$, and $G(\delta; x) = F(\delta; x)$, (5) simplifies to

$$\rho v(x, l[x]) = 2\theta m(x) \left(u - \frac{1}{2} \int_0^{2u} \delta dF(\delta; x) \right) - (k(l[x]) + r(x) l[x]). \qquad (15)$$

In spatial equilibrium, $v(x, l[x]) = w$ for all $x \in [-\overline{x}, \overline{x}]$; i.e.,

$$\frac{dv(x, l[x])}{dx} = 0.$$

Differentiating, while making use of (6), (10), and (13) yields

$$\frac{\partial r(x)}{\partial x} l(x) = \theta \{[H(x+2u) - H(x)] - [H(x) - H(x-2u)]\}. \qquad (16)$$

The LHS of the equation denotes how rental expenditures vary across locations.[6] In spatial equilibrium, any differences in rental expenditure are matched by differences in the payoff of search across locations. To see how the latter differences are governed by the RHS of (16), notice that by (15), the payoff of search can vary as a result of a

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[6] More precisely, it denotes how the the sum $k(l[x]) + r(x) l(x)$ changes with x. However, by the Envelope theorem, if the land input is chosen optimally, it reduces to just the LHS of (16).

change in location if a new matching rate and/or a new expected trade cost accompany the change in location. Now suppose that the search area has been expanded. There will then be a higher expected utility of consumption net of a given average trade cost, as measured by

$$2\theta \times \partial m/\partial x \times \left(u - \frac{1}{2}\int \delta dF\left(\delta; x\right)\right). \tag{17}$$

Meanwhile, the average search cost also changes and does so in two ways. First, at the initial location, if the search area has been expanded as a result of the change in location, the average trade cost for the agent will increase by the amount

$$2\theta m \times \frac{1}{2}\left(\partial\left[\int \delta dF\left(\delta; x\right)\right]/\partial\left[x-2u\right] + \partial\left[\int \delta dF\left(\delta; x\right)\right]/\partial\left[x+2u\right]\right),$$

holding x constant. It turns out that this increase in the expected trade cost just offsets the increase in the expected net utility of consumption with a given expected trade cost – the amount in (17). The coincidence can be understood as an instance of the Envelope theorem. If an agent's search area is chosen optimally, any effects on the payoff of search due to a change in the search area induced by a change in location vanish at the margin. The change in location, however, also causes the expected trade cost to vary as there is now a new location from which to direct the search, as measured by

$$2\theta m \times \partial\left[\frac{1}{2}\int \delta dF\left(\delta; x\right)\right]/\partial x,$$

holding constant the end points of the search area. The RHS of (16) denotes just this effect – an amount proportional to the difference between the population mass residing to the right of the agent's location and that to the left. If this difference is positive, a move to the right side is a move closer to the majority of one's potential trading partners, after which it is less costly to trade on average.

To proceed, notice that the rent gradient on the LHS of (16) is by (13):

$$\frac{\partial r\left(x\right)}{\partial x} = -\frac{\partial^2 k\left(l\left[x\right]\right)}{\partial l^2}\frac{\partial l\left(x\right)}{\partial x}.$$

In spatial equilibrium $h(x) = 1/l(x)$,

$$\frac{\partial r(x)}{\partial x} = \frac{\partial^2 k\left(h(x)^{-1}\right)}{\partial l^2} \frac{\partial h(x)}{\partial x} h(x)^{-2}. \tag{18}$$

A particular functional form for the $k(l)$ function that helps deliver an analytical solution to (16) is

$$k(l) = \frac{\kappa}{2} l^{-1} \tag{19}$$

for some parameter $\kappa > 0$. Then (18) becomes

$$\frac{\partial r(x)}{\partial x} = \kappa \frac{\partial h(x)}{\partial x} h(x).$$

Substituting the result into (16), we obtain

$$\frac{\partial h(x)}{\partial x} = -\frac{\theta}{\kappa}\left(2H(x) - H(x-2u) - H(x+2u)\right). \tag{20}$$

Given that $h(x) = \partial H(x)/\partial x$, (20) is a second-order differential equation in $H(x)$ with fixed leads and lags. For the solution to constitute the equilibrium location distribution, it must first meet the restrictions in (1). An additional restriction comes from (14). With the functional form assumption in (19) and that $h(x) = 1/l(x)$, the condition is

$$h(-\overline{x}) = h(\overline{x}) = \sqrt{\frac{2\overline{r}}{\kappa}}. \tag{21}$$

Without first solving (20), it is possible to establish:

Lemma 2 *If $H(x)$ is symmetric around $x = 0$, then for x that satisfies both* (11) *and* (12), $\partial h(x)/\partial|x| < 0$.

Proof. If both (11) and (12) hold, $H(x-2u) = 0$ and $H(x+2u) = n$. Then (20) becomes

$$\frac{\partial h(x)}{\partial x} = -\frac{\theta}{\kappa}\left[2H(x) - n\right] \underset{>}{<} 0 \Leftrightarrow x \underset{<}{>} 0,$$

because by symmetry,

$$H(x) \gtrless \frac{n}{2} \Leftrightarrow x \gtrless 0.$$

■

Think of two locations x and x', where $|x| < |x'|$. If for either location, the two conditions (11) and (12) apply, then agents at the location shall find it optimal to search from one to the other end of the regional economy for a trading partner. The occupants of x and x' thus face the same matching rate. However, the distance on average that they must cover to trade is not the same. Lemma 2 is based on the premise that as long as $H(x)$ is symmetric around $x = 0$, the distance on average that separates an agent at x and *all others* is increasing in $|x|$. Occupants of the more central location then trade at a lower cost on average than occupants of the more distant location. If the latter location is less desirable, then follows a lower land rent, a higher land demand from the occupants of the location in response, and ultimately a lower density.

Now if the conditions of the lemma are satisfied for all $x \in [-\overline{x}, \overline{x}]$, by Lemma 2, $H(x)$ is a unimodal distribution in which the density decreases everywhere with increasing distance from the center. The two conditions (11) and (12) hold for all $x \in [-\overline{x}, \overline{x}]$ if

$$\overline{x} < u. \tag{22}$$

In a compact region, everybody will choose to search over all locations in the region. In this environment, the advantage of a particular location is entirely determined by how far on average its occupants must travel to execute a trade. If $H(x)$ is symmetric around $x = 0$, central locations are more advantageous and thus more densely populated. The following lemma presents the result of resolving the equilibrium location distribution H for values of $\overline{x}$ that satisfy (22).

Lemma 3 *Write*

$$\overline{x} = \sqrt{\frac{\kappa}{2\theta}} \arcsin\left(\sqrt{\frac{\theta n^2}{\theta n^2 + 4\overline{r}}}\right). \tag{23}$$

If $\overline{x} < u$*, the equilibrium location distribution is given by*

$$H(x) = \frac{n}{2} + \frac{\sqrt{n^2 + 4\overline{r}/\theta}}{2} \sin \omega_1 x \tag{24}$$

for $x \in [-\overline{x}, \overline{x}]$, *where* $\omega_1 = \sqrt{2\theta/\kappa}$. $H(x)$ *is symmetric around* $x = 0$.

Proof. See the proof of Lemma 7 in the appendix.

Lemma 4 *In* (23) *and* (24), $\partial\overline{x}/\partial n > 0$ *and* $\partial h(x)/\partial n > 0$, *respectively, for* $x \in (-\overline{x}, \overline{x})$.

Lemma 4 shows that first, not surprisingly, a more populous region must be a physically larger region. More interestingly, the larger region is also a denser region. That is, the physical expansion is less than in proportion to the population increase, while there is a higher density at each interior location to accommodate the additional population. Such comparative statics mirror the analogous result in the textbook urban model, in which the increases in land rent and therefore density at each interior location that follow a given population increase are due to the greater saving in commuting expense in relation to the costlier commuting incurred by households at a more distant border. In the present model, the increases in land rent and density at each interior location are due to the greater saving in expected trade cost in relation to the higher expected trade cost incurred by agents at a more distant border.

For sufficiently large n, the condition $\overline{x} < u$ may fail. In that case, (11) and (12) would no longer hold for all $x \in [-\overline{x}, \overline{x}]$.[7] Figure 1 illustrates how agents' search areas

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[7] The LHS of (23) is bounded as n increases indefinitely. For certain parameter configurations, the condition can always be met for any values of n. In this case, at the limit, the decrease in land usage per agent compensates one-for-one for the increase in land demand due to the population

may depend on their locations, where

$$u \leq \overline{x} < 2u, \tag{25}$$

$$2u \leq \overline{x} < 3u, \tag{26}$$

respectively. In a region whose border $\overline{x}$ lies between the two bounds in (25), there remains an interval $(-\widetilde{x}, \widetilde{x})$ for $\widetilde{x} = 2u - \overline{x} \in (0, \overline{x}]$, over which the two inequalities (11) and (12) continue to hold, to which Lemma 2 applies. However, the search areas for agents at $x \in [\widetilde{x}, \overline{x}]$ stop short of reaching the left border $-\overline{x}$, and for agents at $x \in [-\overline{x}, -\widetilde{x}]$, the search areas end at some point before the right border $\overline{x}$ is reached. Along these two intervals, the symmetry of $H(x)$ does not suffice to sign $\partial h(x)/\partial x$. In an even larger region, such as where $\overline{x}$ lies between the two bounds in (26), everyone searches only within a subset of $[-\overline{x}, \overline{x}]$. In that case, the symmetry of $H(x)$ cannot be used to sign $\partial h(x)/\partial x$ for any interval within $[-\overline{x}, \overline{x}]$. A general result on $H(x)$ that can be obtained, irrespective of the value of $\overline{x}$, is as follows.

Lemma 5 *For x sufficiently close to the right border $\overline{x}$, $\partial h(x)/\partial x < 0$, and for x sufficiently close to the left border $-\overline{x}$, $\partial h(x)/\partial x > 0$.*

Proof. At $x = \overline{x}$, $H(x + 2u) = H(x) = n$, by (20),

$$\frac{\partial h(\overline{x})}{\partial x} = -\frac{\theta}{\kappa}(n - H(\overline{x} - 2u)) < 0.$$

By continuity, $\partial h(x)/\partial x < 0$ remains satisfied for x sufficiently close to $\overline{x}$. At $x = -\overline{x}$, $H(x - 2u) = H(x) = 0$, by (20),

$$\frac{\partial h(-\overline{x})}{\partial x} = \frac{\theta}{\kappa}(H(x + 2u)) > 0.$$

By continuity, $\partial h(x)/\partial x > 0$ remains satisfied for x sufficiently close to $-\overline{x}$. ■

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increase, which leaves the borders of the region unchanged. Meanwhile, there is an arbitrarily large density at each location. This peculiar feature of the model is a result of the particular functional form assumption in (19).

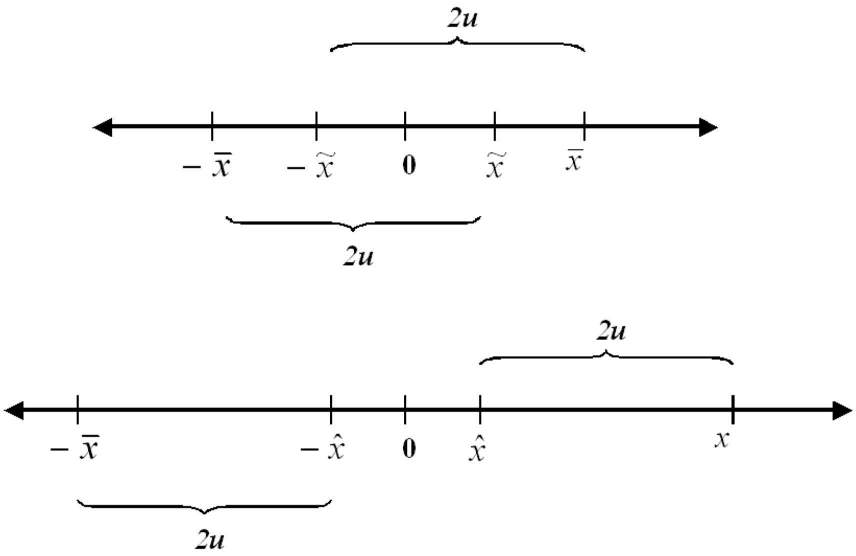

FIG. 1. Locations and Search Areas

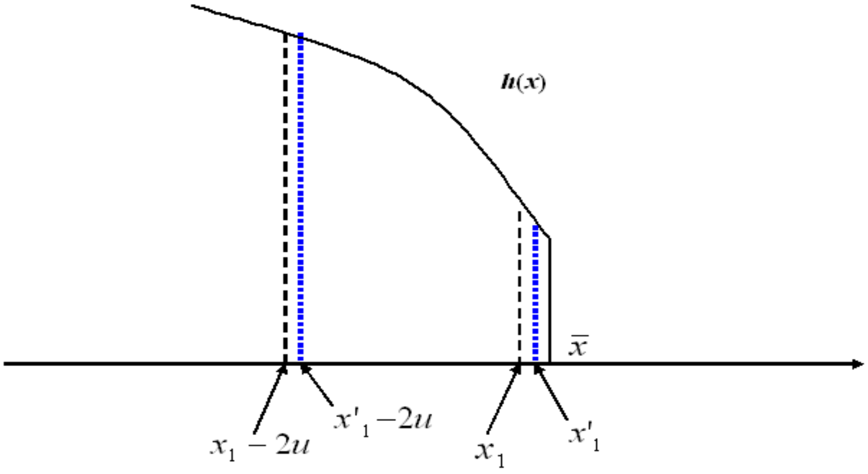

FIG. 2. Locations, Matching Rates, and Expected Trade Costs

Figure 2 illustrates how an agent at some location x_1 near the right border $\overline{x}$ moves to a less desirable location when she moves a bit towards $\overline{x}$. First, say $x_1 - 2u > -\overline{x}$ but $x_1 + 2u > \overline{x}$. That is, initially the agent's search ceases before she reaches the left border $-\overline{x}$ of the region, but her search to the right side takes her all the way to the right border $\overline{x}$. By moving a bit to the right side of x_1, the agent's search can now cover only a smaller area. Potential trading partners are fewer and the matching rate is lower. Still, it is not true that the agent must be moving to a less desirable location. The new location is necessarily less desirable only if searching from the location is also associated with trading at a higher cost on average. Now recall that a move to the right takes the given agent closer to the majority of her potential trading partners and thereby lowers the expected trade cost if and only if

$$[H(x+2u) - H(x)] - [H(x) - H(x-2u)] > 0.$$

For a location x_1 arbitrarily close to the right border of the regional economy $\overline{x}$, $H(x_1) \simeq n$, and the above must hold only in reverse.

To establish how the density changes over space in other locations, one must explore the properties of the solution of $H(x)$. Lemmas 7 and 8 in the appendix present the analytical solutions for $H(x)$ for values of $\overline{x}$ that lie within the bounds in (25) and (26), respectively. As expected, the solutions do turn out to be symmetric functions around $x = 0$. The other properties of the solutions are difficult to establish analytically because of the solutions' complexities. I should proceed by studying a few numerical examples shown in Figure 3. Each curve in the figure corresponds to a different value of n, where at $n = 0.1$ and 0.35, the respective values of $\overline{x}$ satisfy (22), at $n = 0.5$ and 0.9, the respective values of $\overline{x}$ satisfy (25), and at $n = 1.2$ and 1.6, the respective values of $\overline{x}$ satisfy (26).[8]

Evidently, the conclusion of Lemma 2 that $h(x)$ decreases everywhere in $|x|$ remains

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[8] The other parameters are set equal to $u = 0.5$, $\kappa = 1.5$, $\theta = 1$, and $\overline{r} = 0.1$.

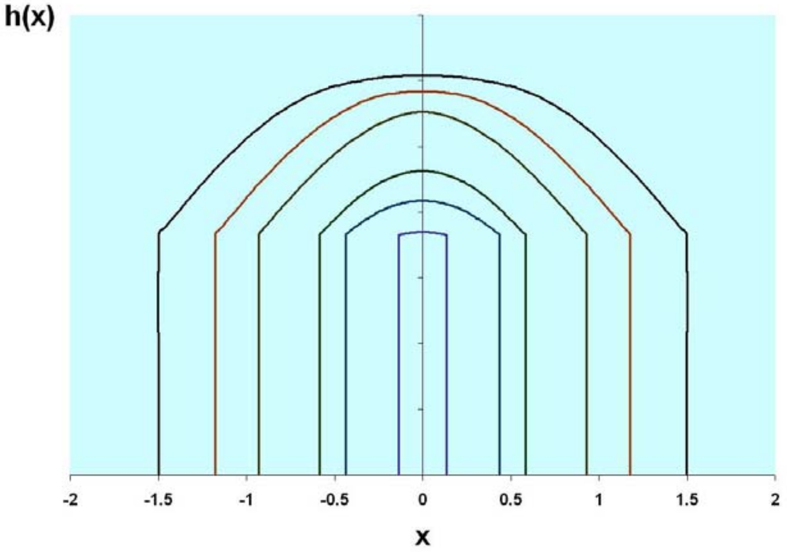

FIG. 3. Density $h(x)$: the curves, from the innermost to the outermost, are for $n = 0.1$, 0.35, 0.5, 0.9, 1.2, and 1.6, respectively.

valid even though the force underlying the lemma may not be applicable. What drives such a tendency seems to be the unraveling of the force underlying Lemma 5. The argument runs as follows. Fix a location x_1 arbitrarily close to $\overline{x}$. By Lemma 5, $\partial h(x)/\partial x < 0$ for $x \in [x_1, \overline{x}]$. Next consider a location x_2 arbitrarily close to the left side of x_1. If $\partial h(x)/\partial x < 0$ for $x \in [x_1, \overline{x}]$, then around x_2 are more densely populated locations than those around x_1. Occupants of x_2 then travel a shorter distance on average to carry out a trade compared to their x_1 counterparts. Now that x_2 is a more desirable location, it should be more densely populated than x_1. For the same reason, a location x_3 arbitrarily close to the left side of x_2 should be a better location than x_2 and thus more densely populated, given that $\partial h(x)/\partial x < 0$ for $x \in [x_2, \overline{x}]$. Hence, even for larger regions, in which no one chooses to search from one to the other border, $h(x)$ remains unimodal and decreases everywhere with increasing distance from the center in spatial equilibrium. Such a tendency is the result of how the advantage of one location depends on and helps determine the advantages of neighboring locations.

3. POPULATION, DENSITY, AND PRODUCTIVITY

In this section, I make use of the properties of spatial equilibrium to study how an agent's productivity depends on the densities of the locations around which she resides. In particular, if on average an agent trades $2\theta m(x)$ times per unit of time, then the agent's rate of output must be just $2\theta m(x)$. However, such a turnover rate is at the expense of $\delta/2$ each time the agent trades and produces. With u denoting the value of a unit of output, the agent's productivity may thus be defined as

$$y(x) = 2\theta m(x)\left(u - \frac{1}{2}\int_0^{2u} \delta dF(\delta; x)\right),$$

which is simply the flow payoff of search.

Proposition 1 *Across locations within the regional economy,*

$$\frac{\partial y(x)}{\partial x} = \kappa \frac{\partial h(x)}{\partial x};$$

i.e.,

$$\frac{\partial y(x)}{\partial x} \gtrless 0 \Leftrightarrow \frac{\partial h(x)}{\partial x} \gtrless 0.$$

Hence, agents at the most dense locations are the most productive.

Proof. By virtue of (16), (18), and (20). ■

The link between density and productivity is immediate once it is recognized that a location may end up being more densely populated only because its occupants can trade more often and at a lower cost on average. However, agents at such a location are also liable to pay higher rents. In response, they hire fewer land inputs while incurring higher non-land expenditures in inventory management. An alternative and perhaps more appropriate definition of productivity is then

$$\widetilde{y}(x) = y(x) - (k(l[x]) + \overline{r}l[x]), \tag{27}$$

where the expenditure for inventory management is subtracted from $y(x)$. In this definition, a unit of land is valued at the opportunity cost of rural land but not the actual land rent that the agent pays, the amount $r(x)$. The rationale is that location rents – the differences between $r(x)$ and $\overline{r}$ for $|x| < \overline{x}$ represent pure transfers between agents in the regional economy and absentee landlords.

The amount $k(l[x]) + \overline{r}l(x)$ in (27) may be thought of as a location cost, which is lowest for agents residing right at the borders as these agents choose l just to minimize $k(l) + \overline{r}l$, whereas agents at interior locations choose l to minimize $k(l) + r(x)l$ with $r(x) > \overline{r}$. In this manner, the higher productivity that agents at denser locations attain is at the expense of a higher location cost.

Proposition 2 *Across locations within the regional economy,*

$$\frac{\partial \widetilde{y}(x)}{\partial x} \gtrless 0 \Leftrightarrow \frac{\partial y(x)}{\partial x} \gtrless 0.$$

Proof. Differentiating with $l(x) = 1/h(x)$. ■

Proposition 2 is a direct consequence of spatial equilibrium, in which $r(x)$ varies to equalize

$$\rho v(x, l[x]) = y(x) - (k(l[x]) + r(x)l[x]).$$

If $v(x, l[x])$ is equalized across $x \in [-\overline{x}, \overline{x}]$, then there must be higher $\widetilde{y}(x)$ in the denser interior locations with $r(x) > \overline{r}$ for $|x| < \overline{x}$. In sum, Propositions 1 and 2 together show that even after the higher location costs for denser locations are taken into account, the positive relation between density and productivity remains.

Propositions 1 and 2 concern the relation between density and productivity within a region with a given population. For cross-regional comparisons, Proposition 3 establishes that:

Proposition 3 *As a function of* n*, for all* $x \in [-\overline{x}, \overline{x}]$ *,* $\partial y(x) / \partial n > 0$ *if* $\partial h(x) / \partial n > 0$ *for all* $x \in (-\overline{x}, \overline{x})$ *.*

Proof. Write $y(x)$ instead as

$$y(x) = 2\theta \int_{x-2u}^{x+2u} \left(u - \frac{1}{2} |x - s| \right) dH(s) . \tag{28}$$

The lemma then is guaranteed to hold as long as there is a higher density at each x. ■

Proposition 3 is due to the fact that if under optimal search, the agent should expand the search to any locations at which a trade with the occupants of the locations will yield a positive match surplus, then the agent must be better off when those locations are more densely populated. The condition of the proposition, by Lemma 4, must hold for smaller regions, where $\overline{x} < u$. To verify that the condition holds for all values of $\overline{x}$ is difficult, given that the analytical solutions for $H(x)$ for larger regions, as presented in Lemmas 7 and 8 in the appendix, are rather complicated expressions. The numerical examples shown in Figure 3 do suggest that the condition should hold for all values of $\overline{x}$. After all, the validity of the argument for such a tendency, which is that as the region expands horizontally, each interior location will experience an increase in location rent, should not depend on the initial value of $\overline{x}$.

Given that any population increase should cause the region to expand in size, agents located along border locations will certainly benefit by being able to enlarge their search areas. To follow will be an increase in matching rates and a decline in waiting times between the production of successive units. More subtly, agents at central locations will also be able to search at a higher matching rate, even though the physical expansion of the region might not lead to any physical expansion in their search areas in the case that the end points of the areas over which they search were not initially constrained by the boundaries of the region. Such a tendency is the result of the increase in density at each interior location caused by the population increase

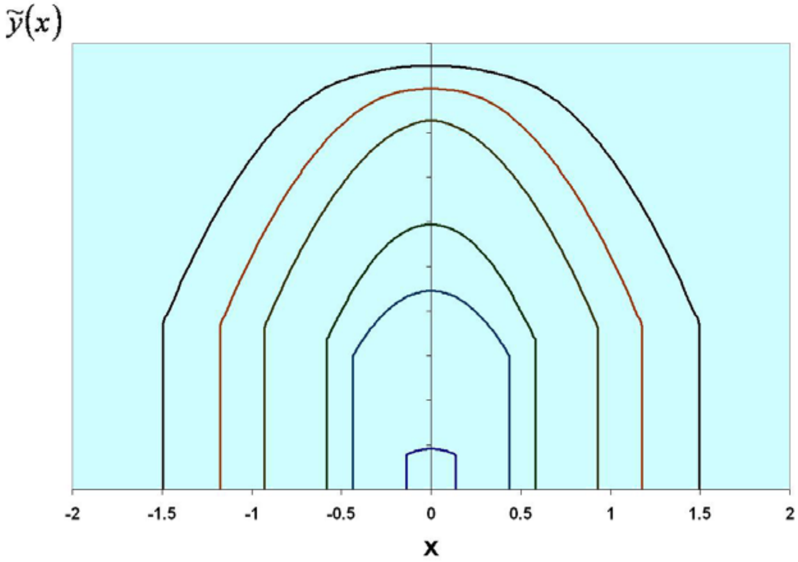

FIG. 4. Productivity $\widetilde{y}(x)$: the curves, from the innermost to the outermost, are for $n = 0.1$, 0.35, 0.5, 0.9, 1.2, and 1.6, respectively.

– a thick market externality that arises endogenously in a spatial model.

What guarantees that productivity must rise for everybody is that agents choose their search areas optimally. If instead each agent's search is random and covers all locations in the regional economy, then a sufficiently large physical expansion can lower productivity if the increases in distance-related trade expenses overwhelm the benefits of higher matching rates. In (28) , for instance, if the search were from one to the other end of the region rather than ceasing at where $u - \delta/2$ remains non-negative, a sufficiently large physical expansion could certainly lower $y(x)$. Where agents choose their search areas optimally, they will just choose not to travel to locations that are too costly to reach.

A given population increase leads to an increase in not only $y(x)$ but also $k(l[x]) + \overline{r}l(x)$, as the increase in land rent leads to a decrease in land input. In the present model, the increase in location cost accompanying the increase in density represents

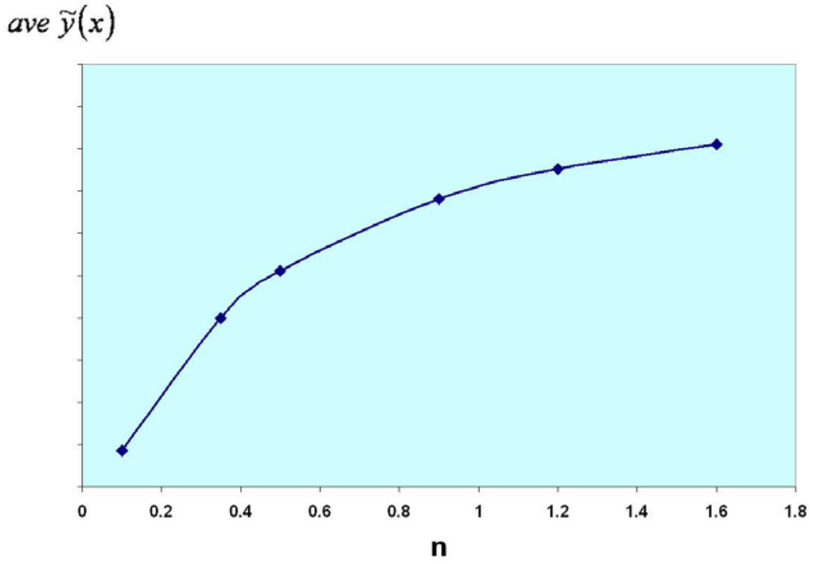

Fig. 5. Average Productivity

a centrifugal force that counteracts the positive effect of agglomeration. Figure 4 shows that for the same set of parameter values as those in Figure 3, $\widetilde{y}(x)$ remains an increasing function of n for each location. Figure 5 shows how the average net productivity, defined as

$$average\ \widetilde{y} = \frac{\int_{-\overline{x}}^{\overline{x}} \widetilde{y}(x)\, dH(x)}{n},$$

increases with n for the six values of n in Figure 4. The tendencies shown in the two figures though are probably specific to the functional form assumption on $k(l)$ in (19). With certain alternative functional form assumptions, the average net productivity could well reach its maximum at some finite value of n.[9]

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[9] While the conjecture should be verified by solving the model with some alternative $k(l)$ function, perhaps one where k increases more rapidly with a given decline in l, the functional form in (19) seems to be the only functional form that helps turn (16) into a linear and tractable differential equation.

To summarize, by virtue of being able to trade more frequently and at a lower cost on average, within a given region, agents at denser locations should produce at higher rates of output, even after the higher location costs at such locations are taken into account. Across regions, a positive relation should exist between population and density on the one hand and productivity on the other hand, where agents in the denser, larger, and more populous region are able to trade more often.

4. SOCIAL OPTIMUM

Clearly, given a location distribution $H(x)$, the decisions of private agents to extend their search to any locations at which a trade with the occupants of the locations will yield a positive match surplus are not just privately but also socially optimal. What is less clear is whether agents' choices of land inputs coincide with those made by a planner whose objective is to maximize aggregate net output,

$$\widetilde{Y} = \int_{-\overline{x}}^{\overline{x}} \widetilde{y}(x)\, dH(x),$$

with respect to $\overline{x}$ and $H(x)$.

Lemma 6 *In the planner's allocation, the border density coincides with the equilibrium border density, given by* (21)*, while the rate at which the optimum density changes over space is given by*

$$\frac{\partial h(x)}{\partial x} = -\frac{2\theta}{\kappa}\left(2H(x) - H(x-2u) - H(x+2u)\right). \tag{29}$$

Proof. In the appendix.

Notice that the optimality condition (29) is otherwise identical to the equilibrium condition (20), except that θ in the latter is replaced by 2θ in the former. Then given that the identical border densities in the equilibrium and optimum allocations are independent of θ, the optimum location distribution may simply be obtained by replacing θ in the equilibrium location distributions in Lemmas 3, 7, and 8 with 2θ. In this case, with the same candidate H function, the optimum density increases towards its maximum at the regional center at twice the rate that the equilibrium density does. As a result, the optimum distribution should be more concentrated than the equilibrium distribution. Figure 6 illustrates an example of this tendency with $n = 1.6$.[10] In a free market economy then, the region is too large and too sparsely populated relative to the optimum. The conclusion is similar to the result derived in Beckmann (1977), in which households ignore the effects of their space demand choices on the traveling expenses of others. In the Beckmann model, when a household occupies a smaller lot, the city becomes denser and more compact, and others benefit by travelling shorter distances to reach the household. In the present model, in addition to the effect uncovered in Beckmann, there is the additional externality that in a denser region, each agent can match with another and produce more frequently.

Recall that the RHS of the equilibrium condition (20) denotes how the flow payoff of search as given by the first term of (15) varies over space. Hence, if this flow payoff increases by a factor of two, then (20) will coincide with the optimum condition (29). One alternative to restore optimality then is to subsidize each agent by an amount equal to her share of the match surplus each time that she trades. Intuitively, when search becomes more lucrative and if search is most effective when it is directed from central locations, the subsidy scheme helps increase the advantages of central locations relative to border ones. The rent gradient steepens as a result, inducing centrally located agents to conserve land consumption to just about the right amount. The information requirement to implement such a subsidy scheme can be hard to meet in practice, though, if match surplus is private information.

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[10] The other parameters are set as those in Figure 3. While the equilibrium border $\overline{x}$ lies between the bounds in (26), the optimum border lies between the bounds in (25).

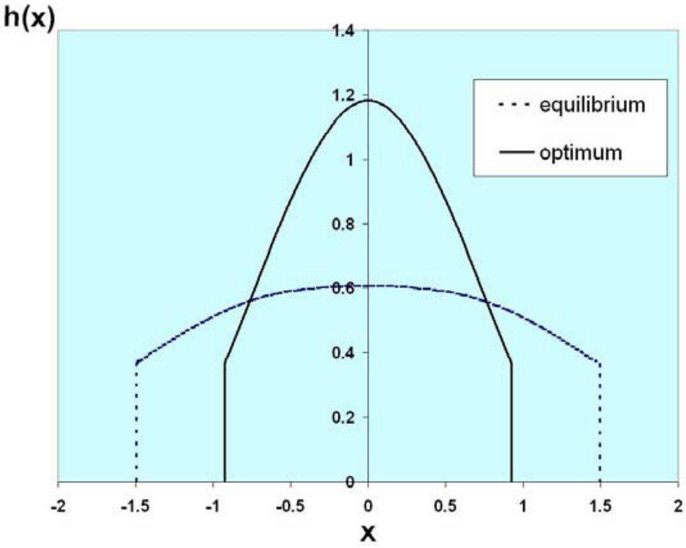

FIG. 6. Optimum and Equilibrium Location Distributions

Given that the inefficiency is due to excessive land consumption, perhaps a more direct means to restore optimality is to tax land consumption. Taxing land consumption to induce efficient behavior is not necessarily straightforward, for it is not that the level of rent is too low in equilibrium but rather that the rent gradient is not steep enough. For instance, in Figure 6, if the border densities in the equilibrium and optimum allocations coincide, land consumption right at the border should not be taxed at all. In particular, the optimum tax rate, starting at zero at the regional border, should gradually increase towards the center.

A simpler alternative is not to tax but to subsidize land consumption. More accurately, if agents are liable to pay only one half of the flow expenditure for inventory management, the amount $k(l)+r(x)l$, then the LHS of (20) becomes $1/2\times\partial h(x)/\partial x$, turning the condition into the optimum condition (29). Meanwhile, the same condition (13) governs agents' choices of land consumption, for a given unit land rent function, in the absence of any distortion in the tradeoff between the two kinds of inputs. Then there must be the same border densities as defined by (14). Hence, to restore optimality is to subsidize one half of agents' expenditures for non-land and land inputs in inventory management. If such expenditures are thought of as the costs of search, the subsidy scheme works by increasing the payoff of search *relative* to the cost of search by two times – which is what 100% subsidies on match surpluses would help obtain.

5. DISCUSSION

The spatial structure Throughout the analysis, an important assumption is that the regional economy is situated on a linear space. It goes without saying that in a one-dimensional circular space, no one location is more advantageous than another location. Then in spatial equilibrium, there can only be a uniform distribution of agents over space. Specifically, if not for the constraints in (1), solving (20) would simply yield a uniformly distributed H function. In the real world, of course, cities and regions do have boundaries. That the analysis does not apply to a one-dimensional circular space should not be a serious concern. A more legitimate question is how the analysis fares if the region is a circle on a two-dimensional plane, which is certainly a much better approximation of reality than a one-dimensional linear space. A complete answer awaits a full-fledged analysis. Nevertheless, it seems unlikely that the major qualitative findings: (i) a negative relation exists between a location's density and its distance to the regional center and (ii) a positive relation exists between density and productivity within and between regions, would be overturned. If, as in Lucas (2000), the analysis is restricted to an equilibrium in which a location is indexed only by its distance from the regional center without reference to its angle with respect to the origin of the two-dimensional plane, locations differ from one another only to the extent that they are situated from the center at various distances. Effectively, then, a problem in a two-dimensional space is reduced to a problem in a one-dimensional linear space.

More specifically, if the present model is embedded in a two-dimensional space, then an agent's search area still comprises all locations up to a distance of $2u$ from the agent's location. One advantage of a central location over a border one then is that the search area of an occupant of the former can be a full circle, whereas the search area of an occupant of the latter is constrained by the borders of the region in various directions. Moreover, in the case where the two agents both choose to search over all locations in the two-dimensional circle, the more centrally located agent tends to trade at a lower cost as there is a shorter distance on average separating her from others in the region as long as the distribution of agents is symmetric around the origin of the two-dimensional plane.

Trade cost, search cost, and distance In reality, trade costs are substantial. Aside from costs arising from policy barriers, retail and wholesale costs alone can be equivalent to a 55% ad valorem tax (Anderson and van Wincoop 2004). One obvious reason why it is more costly for one to trade with someone in a remote location than with someone nearby is that it is more costly to ship goods over a longer distance. In modern times, however, the costs of communicating product designs, contract negotiation and enforcement, and other related expenditures are often more important components of trade costs than the cost of shipping goods physically from one place to another. Still, there is good reason why the former category of trade costs should also increase with the physical distance between trading parties. In many instances, in a technologically advanced economy, much of the information to be transmitted for successful exchanges cannot be easily codified, and repeated face-to-face communication is crucial for the transmission of such tacit information (David 1992). It is not hard then to envisage that trading with someone in a far-flung location is more costly than trading with one's neighbor.

Nevertheless, where the goods to be bought and sold are standardized commodities, the expenditure for completing a trade may be dwarfed by the expenditure for searching for where the goods is available. In this case, the more relevant spatial friction is a distance-related search cost. To model such spatial friction, it is more appropriate to assume that the search cost that an agent incurs is a function of the distance that she travels in searching for a trading partner[11] (Schwartz 1976, Seater 1979) than to assume that trade cost is a function of the distance separating the given agent and her eventual trading partner. In this model, an agent should search out a new location if the increase in the search cost does not exceed the benefit obtained from an increase in the matching rate. The main difficulty in analyzing this model is that whereas the latter increase depends on the density of the location in question, the former increase does not if the marginal search cost is in the main a function of the distance that the search has already covered. In such a case, agents' optimal search areas are no longer bounded by a certain model parameter but rather are functions of the equilibrium location distribution, turning the analogue of (20) into a differential equation with endogenous leads and lags, the solution of which remains unknown (Boucekkine et al. 1997).[12]

Negative thick market externality The analysis in this paper has emphasized the merits of a thick market. However there are situations in which certain players in a market are hurt by an expansion of the market. For instance, when prices fall

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[11] In a way that can be thought of as a space economy-based microeconomic foundation of the assumption in Mortensen (1982) that agents trade a lower search cost for a higher matching rate.

[12] In the model of this paper, the downside of extending the search to a new location is the increase in the expected trade cost, which is also a function of the density of the new location. In case $\beta(m) = \theta m$, the effects of the density of the new location on the benefits and costs of extending the search to the location just offset each other. The remaining consideration is whether trading with the occupants of the location will yield a positive surplus.

because of the more intense competition that follows from an increase in the number of entrants in a market, the incumbents are surely worse off. Nevertheless, even in this case, firms can benefit from the presence of competitors, as argued by Rotemberg and Saloner (2000), if the competition serves to resolve the hold-up problem by relieving factor suppliers from the concern that their sunk investment will fail to yield any returns at all in a market that can support only a single producer.

6. CONCLUDING REMARKS

In models of decentralized and bilateral trade, it is often assumed that there is more rapid turnover in “thicker” markets. Thick market externalities have a distinct spatial favor as physical proximity promotes interaction and facilitates trade. The occupants of more central locations can search among a larger set of potential trading partners and at a lower cost compared to their less centrally located counterparts, which enables the former to achieve a faster rate of turnover in output. In spatial equilibrium, there must exist a positive relation between density and productivity. Moreover, where more populous regions are denser regions, at least along some given range of population, workers in such locales can produce at a higher rate of output because they have more abundant opportunities for exchange compared to those in less populous and less dense regions.

A defining characteristic of big cities arguably is the fast-paced and hectic environment. More formally, Rosenthal and Strange (2008) find that professional workers in big cities work significantly longer hours than their counterparts in less populous locales. The analysis in this paper suggests one explanation for why big cities are such busy places: the abundance of trading opportunities compels firms and their workers to produce at a more rapid rate to keep pace with the faster rate of turnover.

APPENDIX

A1. Analytical solutions for $H(x)$

Lemma 7 *Given C_0, D_1, and D_2, define*

$$H_2(x) = \frac{n}{3} + D_1(\sin\omega_0 x + \sin\omega_0(x+2u)) + D_2(\cos\omega_2 x + \cos\omega_2(x+2u)),$$

$$H_3(x) = \frac{n}{2} + C_0 \sin\omega_1 x,$$

$$H_4(x) = \frac{2n}{3} + D_1(\sin\omega_0 x + \sin\omega_0(x-2u)) - D_2(\cos\omega_2 x + \cos\omega_2(x-2u)),$$

where $\omega_0 = \sqrt{\theta/\kappa}$, $\omega_1 = \omega_0\sqrt{2}$, and $\omega_2 = \omega_0\sqrt{3}$. Let $\{\overline{x}, C_0, D_1, D_2\}$ solve the system $(40)-(43)$. *If the solution of $\overline{x}$ thus obtained satisfies*

$$u \le \overline{x} < 2u, \tag{30}$$

then for $\widetilde{x} = 2u - \overline{x}$, the equilibrium location distribution is given by

$$H(x) = \begin{cases} H_2(x) & x \in [-\overline{x}, -\widetilde{x}] \\ H_3(x) & x \in (-\widetilde{x}, \widetilde{x}) \\ H_4(x) & x \in [\widetilde{x}, \overline{x}] \end{cases} . \tag{31}$$

Proof.

Partitions Under (30),

$$\overline{x} \ge \widetilde{x} > 0.$$

Then $[-\overline{x}, \overline{x}]$ may be partitioned into three intervals. In the first interval, for $x \in [-\overline{x}, -\widetilde{x}]$,

$$x - 2u < -\overline{x}; \quad \widetilde{x} \le x + 2u \le \overline{x}.$$

In the second interval, for $x \in (-\widetilde{x}, \widetilde{x})$,

$$x - 2u < -\overline{x}; \quad x + 2u > \overline{x}.$$

In the third interval, for $x \in [\widetilde{x}, \overline{x}]$,

$$-\overline{x} \leq x - 2u \leq -\widetilde{x}; \quad x + 2u > \overline{x}.$$

The system of equations With the restrictions in (1) and the division of $H(x)$ in (31), for x in the three intervals, respectively, (20) becomes

$$\frac{\partial h_2(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_2(x) - H_4(x+2u)\right] = 0, \tag{32}$$

$$\frac{\partial h_3(x)}{\partial x} + \frac{2\theta}{\kappa}H_3(x) = \frac{\theta}{\kappa}n. \tag{33}$$

$$\frac{\partial h_4(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_4(x) - H_2(x-2u)\right] = \frac{\theta}{\kappa}n. \tag{34}$$

Eq. (33) is a stand-alone second-order differential equation in H_3, whereas (32) and (34) constitute a system of two second-order differential equations in H_2 and H_4.

The solution We first proceed to solve (33). Assuming a constant particular integral and solving the equation yields the solution $H_3 = n/2$. Next solve the complementary function via the trial solution $H_3(x) = Ce^{\lambda x}$ and add the resulting function to the particular integral; the general solution reads

$$H_3(x) = \frac{n}{2} + C_0 \sin \omega_1 x + C_1 \cos \omega_1 x. \tag{35}$$

For the system (32) and (34), again assuming constant particular integrals and solving the system, the solutions read $\left\{H_2 = \frac{n}{3},\ H_4 = \frac{2n}{3}\right\}$. Next suppose that the complementary functions are respectively,

$$H_2(x) = A\left(e^{\lambda x} + e^{\lambda(x+2u)}\right),$$

$$H_4(x) = B\left(e^{\lambda x} + e^{\lambda(x-2u)}\right).$$

Then the reduced equations of (32) and (34) become, respectively,

$$\left(\lambda^2 + \frac{2\theta}{\kappa}\right)A - \frac{\theta}{\kappa}B = 0, \tag{36}$$

$$-\frac{\theta}{\kappa}A+\left(\lambda^2+\frac{2\theta}{\kappa}\right)B=0. \tag{37}$$

The characteristic equation is

$$\left(\lambda^2+\frac{2\theta}{\kappa}\right)^2-\left(\frac{\theta}{\kappa}\right)^2=0,$$

yielding four imaginary roots: $\lambda_1 = i\omega_0$, $\lambda_2 = -i\omega_0$, $\lambda_3 = i\omega_2$, and $\lambda_4 = -i\omega_2$. Corresponding to the four roots are four pairs of arbitrary constants. By (36) and (37), the arbitrary constants satisfy:

$$\begin{aligned} B_1 &= A_1, \\ B_2 &= A_2, \\ B_3 &= -A_3, \\ B_4 &= -B_4. \end{aligned}$$

Adding the complementary function for H_2 to its particular integral and simplifying, the general solution reads

$$\begin{aligned} H_2(x) &= \frac{n}{3}+D_0(\cos\omega_0 x+\cos\omega_0(x+2u))+D_1(\sin\omega_0 x+\sin\omega_0(x+2u))+ \\ &\quad D_2(\cos\omega_2 x+\cos\omega_2(x+2u))+D_3(\sin\omega_2 x+\sin\omega_2(x+2u)), \end{aligned} \tag{38}$$

where $D_0 = A_1 + A_2$, $D_1 = (A_1 - A_2)i$, $D_2 = A_3 + A_4$, and $D_3 = (A_3 - A_4)i$. Adding the complementary function for H_4 to its particular integral and simplifying, the general solution reads

$$\begin{aligned} H_4(x) &= \frac{2n}{3}+D_0(\cos\omega_0 x+\cos\omega_0(x-2u))+D_1(\sin\omega_0 x+\sin\omega_0(x-2u))- \\ &\quad D_2(\cos\omega_2 x+\cos\omega_2(x-2u))-D_3(\sin\omega_2 x+\sin\omega_2(x-2u)). \end{aligned} \tag{39}$$

The arbitrary constants Eqs. (35), (38), and (39) constitute a complete characterization of $H(x)$ over $x \in [-\overline{x}, \overline{x}]$. Yet to be determined are $\{C_0, C_1, D_0, D_1, D_2, D_3\}$, in addition to $\overline{x}$. The conditions are

$$H_2(-\overline{x})=0, \quad H_4(\overline{x})=n, \tag{40}$$

$$H_2(-\widetilde{x}) = H_3(-\widetilde{x}), \quad H_3(\widetilde{x}) = H_4(\widetilde{x}), \tag{41}$$

$$h_2(-\overline{x}) = \overline{h}, \quad h_4(\overline{x}) = \overline{h}, \tag{42}$$

$$h_2(-\widetilde{x}) = h_3(-\widetilde{x}), \quad h_3(\widetilde{x}) = h_4(\widetilde{x}), \tag{43}$$

where $\overline{h} = \sqrt{2\overline{r}/\kappa}$ is the equilibrium border density.

First, for the four equalities of (40) and (42) to hold all at once, $D_0 = D_3 = 0$. Then only one of the two equations in (40) and (42), respectively, remain an independent restriction. Next, for the four equalities of (41) and (43) to hold all at once, $C_1 = 0$. Then only one of the two equations in (41) and (43), respectively, remain an independent restriction. There are four unknowns: $\{C_0, D_1, D_2, \overline{x}\}$ in four equations. With $D_0 = D_3 = 0$, $H_2(-x) = n - H_4(x)$ and $h_2(-x) = h_4(x)$. With $C_1 = 0$, $H_3(-x) = n - H_3(x)$ and $h_3(-x) = h_3(x)$. That is, H is everywhere symmetric around $x = 0$.

Where $\overline{x} < u$ In this case for all $x \in [-\overline{x}, \overline{x}]$,

$$x - 2u < -\overline{x}; \quad x + 2u > \overline{x}.$$

Thus for all $x \in [-\overline{x}, \overline{x}]$, $H(x)$ satisfies (33), the general solution of which is given by (35). In this case, $\{C_0, C_1\}$ and $\overline{x}$ can be pinned down by the conditions:

$$H_3(-\overline{x}) = 0, \quad H_3(\overline{x}) = n, \tag{44}$$

$$h_3(-\overline{x}) = \overline{h}, \quad h_3(\overline{x}) = \overline{h}. \tag{45}$$

It follows from (45) that $C_1 = 0$. The two remaining variables C_0 and $\overline{x}$ solve

$$C_0 \sin \omega_1 \overline{x} = \frac{n}{2}, \tag{46}$$

$$\omega_1 C_0 \cos \omega_1 \overline{x} = \overline{h}; \tag{47}$$

i.e.,

$$\sin \omega_1 \overline{x} = \frac{n}{2C_0}, \qquad \cos \omega_1 \overline{x} = \frac{\overline{h}}{zC_0}.$$

Then

$$\left(\frac{n}{2C_0}\right)^2 + \left(\frac{\overline{h}}{\omega_1 C_0}\right)^2 = 1;$$

from which (23) and (24) in Lemma 3 follow. ■

Lemma 8 *Given J_0, J_3, and J_5, define*

$$H_1(x) = \frac{n}{4} + J_0 \cos \omega_1 (x + 2u) + J_3 \sin \omega_3 (x + 2u) + J_5 \sin \omega_4 (x + 2u),$$

$$H_3(x) = \frac{n}{2} + \sqrt{2}\left(J_3 \sin \omega_3 x - J_5 \sin \omega_4 x\right),$$

$$H_5(x) = \frac{3n}{4} - J_0 \cos \omega_1 (x - 2u) + J_3 \sin \omega_3 (x - 2u) + J_5 \sin \omega_4 (x - 2u),$$

where $\omega_3 = \omega_0 \sqrt{2 - \sqrt{2}}$ and $\omega_4 = \omega_0 \sqrt{2 + \sqrt{2}}$. Recall the definitions of $H_2(x)$ and $H_4(x)$ in Lemma 7. Let $\{\overline{x}, J_0, J_3, J_5, D_1, D_2\}$ solve the system $(61) - (66)$. If the solution of $\overline{x}$ thus obtained satisfies

$$2u \leq \overline{x} < 3u, \tag{48}$$

then for $\widehat{x} = \overline{x} - 2u$ and $\widehat{x}' = 4u - \overline{x}$, the equilibrium location distribution is given by

$$H(x) = \begin{cases} H_1(x) & x \in [-\overline{x}, -\widehat{x}'] \\ H_2(x) & x \in (-\widehat{x}', -\widehat{x}) \\ H_3(x) & x \in [-\widehat{x}, \widehat{x}] \\ H_4(x) & x \in (\widehat{x}, \widehat{x}') \\ H_5(x) & x \in [\widehat{x}', \overline{x}] \end{cases}. \tag{49}$$

Proof.

Partitions Under (48),

$$\overline{x} > \widehat{x}' > \widehat{x} \geq 0.$$

In this case, $[-\overline{x}, \overline{x}]$ may be partitioned into five intervals. In the first interval, for $x \in [-\overline{x}, -\widehat{x}']$,

$$x - 2u < -\overline{x}; \quad -\widehat{x} \leq x + 2u \leq \widehat{x}.$$

In the second interval, for $x \in (-\widehat{x}', -\widehat{x})$,

$$x - 2u < -\overline{x}; \quad \widehat{x} < x + 2u < \widehat{x}'.$$

In the third interval, for $x \in [-\widehat{x}, \widehat{x}]$,

$$-\overline{x} \leq x - 2u \leq -\widehat{x}'; \quad \widehat{x}' \leq x + 2u \leq \overline{x}.$$

In the fourth interval, for $x \in (\widehat{x}, \widehat{x}')$,

$$-\widehat{x}' < x - 2u < -\widehat{x}; \quad x + 2u > \overline{x}.$$

In the fifth interval, for $x \in [\widehat{x}', \overline{x}]$,

$$-\widehat{x} \leq x - 2u \leq \widehat{x}; \quad x + 2u > \overline{x}.$$

The system of equations Hence, with the restrictions in (1) and the division of $H(x)$ in (49), for x in the five intervals, respectively, (20) becomes

$$\frac{\partial h_1(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_1(x) - H_3(x+2u)\right] = 0, \tag{50}$$

$$\frac{\partial h_2(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_2(x) - H_4(x+2u)\right] = 0, \tag{51}$$

$$\frac{\partial h_3(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_3(x) - H_5(x+2u) - H_1(x-2u)\right] = 0, \tag{52}$$

$$\frac{\partial h_4(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_4(x) - H_2(x-2u)\right] = \frac{\theta}{\kappa}n, \tag{53}$$

$$\frac{\partial h_5(x)}{\partial x} + \frac{\theta}{\kappa}\left[2H_5(x) - H_3(x-2u)\right] = \frac{\theta}{\kappa}n. \tag{54}$$

There are two separate systems: (i) eqs. (50), (52), and (54) in H_1, H_3, and H_5 and (ii) eqs. (51) and (53) in H_2 and H_4.

The solution We proceed to solve the first system. Assume constant particular integrals; the solutions are $\left\{H_1 = \frac{n}{4}, H_3 = \frac{n}{2}, H_5 = \frac{3n}{4}\right\}$. For the complementary functions, assume

$$H_1(x) = Ee^{\mu(x+2u)},$$

$$H_3(x) = Fe^{\mu x},$$

$$H_5(x) = Ge^{\mu(x-2u)}.$$

Substitute the trial solutions into the reduced equations of (50), (52), and (54):

$$\left(\mu^2 + 2\frac{\theta}{\kappa}\right) E - \frac{\theta}{\kappa} F = 0, \tag{55}$$

$$-\frac{\theta}{\kappa} E + \left(\mu^2 + 2\frac{\theta}{\kappa}\right) F - \frac{\theta}{\kappa} G = 0, \tag{56}$$

$$-\frac{\theta}{\kappa} F + \left(\mu^2 + 2\frac{\theta}{\kappa}\right) G = 0. \tag{57}$$

The characteristic equation is

$$\left(\mu^2 + 2\frac{\theta}{\kappa}\right)\left[\left(\mu^2 + 2\frac{\theta}{\kappa}\right)^2 - 2\left(\frac{\theta}{\kappa}\right)^2\right] = 0$$

yielding six imaginary roots: $\mu_1 = i\omega_1$, $\mu_2 = -i\omega_1$, $\mu_3 = i\omega_3$, $\mu_4 = -i\omega_3$, $\mu_5 = i\omega_4$, and $\mu_5 = -i\omega_4$. Corresponding to the six roots are six triplets of arbitrary constants. By $(55) - (57)$, the arbitrary constants satisfy

$$\begin{aligned}
F_1 &= 0, & G_1 &= -E_1; \\
F_2 &= 0, & G_2 &= -E_2; \\
F_3 &= \sqrt{2}E_3, & G_3 &= E_3; \\
F_4 &= \sqrt{2}E_4, & G_4 &= E_4; \\
F_5 &= -\sqrt{2}E_5, & G_5 &= E_5; \\
F_6 &= -\sqrt{2}E_6, & G_6 &= E_6.
\end{aligned}$$

Adding the complementary function for H_1 to its particular integral and simplifying, the general solution reads

$$\begin{aligned} H_1(x) &= \frac{n}{4} + J_0 \cos \omega_1 (x+2u) + J_1 \sin \omega_1 (x+2u) + J_2 \cos \omega_3 (x+2u) \\ &\quad + J_3 \sin \omega_3 (x+2u) + J_4 \cos \omega_4 (x+2u) + J_5 \sin \omega_4 (x+2u), \end{aligned} \tag{58}$$

where $J_0 = E_1 + E_2$, $J_1 = (E_1 - E_2)\, i$, $J_2 = E_3 + E_4$, $J_3 = (E_3 - E_4)\, i$, $J_4 = E_5 + E_6$, and $J_5 = (E_5 - E_6)\, i$. In similar manners,

$$H_3(x) = \frac{n}{2} + \sqrt{2}\left(J_2 \cos \omega_3 x + J_3 \sin \omega_3 x - J_4 \cos \omega_4 x - J_5 \sin \omega_4 x\right), \tag{59}$$

$$\begin{aligned} H_5(x) &= \frac{3n}{4} - J_0 \cos \omega_1 (x-2u) - J_1 \sin \omega_1 (x-2u) + J_2 \cos \omega_3 (x-2u) \\ &\quad + J_3 \sin \omega_3 (x-2u) + J_4 \cos \omega_4 (x-2u) + J_5 \sin \omega_4 (x-2u). \end{aligned} \tag{60}$$

We next move to the system (51) and (53) in H_2 and H_4. But the system is identical to the system (32) and (34), the general solution of which is given by (38) and (39).

The arbitrary constants Eqs. (58) – (60), together with (38) and (39), constitute a complete characterization of $H(x)$ over $x \in [-\overline{x}, \overline{x}]$. Yet to be determined are $\overline{x}$ and $\{J_0, J_1, J_2, J_3, J_4, J_5, D_0, D_1, D_2, D_3\}$. The conditions are

$$H_1(-\overline{x}) = 0, \quad H_5(\overline{x}) = n; \tag{61}$$

$$H_1(-\widehat{x}') = H_2(-\widehat{x}'), \quad H_4(\widehat{x}') = H_5(\widehat{x}'); \tag{62}$$

$$H_2(-\widehat{x}) = H_3(-\widehat{x}), \quad H_3(\widehat{x}) = H_4(\widehat{x}); \tag{63}$$

$$h_1(-\overline{x}) = \overline{h}, \quad h_5(\overline{x}) = \overline{h}. \tag{64}$$

$$h_1(-\widehat{x}') = h_2(-\widehat{x}'), \quad h_4(\widehat{x}') = h_5(\widehat{x}'), \tag{65}$$

$$h_2(-\widehat{x}) = h_3(-\widehat{x}), \quad h_3(\widehat{x}) = h_4(\widehat{x}), \tag{66}$$

First, for the four equalities of (61) and (64) to hold all at once, $J_1 = J_2 = J_4 = 0$. Then for the four equalities of (62) and (65) to hold all at once, $D_0 = D_3 = 0$. With these restrictions, H is everywhere symmetric around 0, which implies that only one equation of each of (61) – (66) is an independent restriction. There are six equations in six unknowns: $\{J_0, J_3, J_5, D_1, D_2, \overline{x}\}$. ■

A2. Proof

Proof of lemma 6 The objective is to maximize $\widetilde{Y}$ subject to $\int_{-\overline{x}}^{\overline{x}} h(x)\,dx = n$. Let

$$\Delta(x) = m(x) \int_0^{2u} \delta dF(\delta; x).$$

Write the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \int_{-\overline{x}}^{\overline{x}} \left(\theta\left[2u\left(H(x+2u) - H(x-2u)\right) - \Delta(x)\right] - \frac{\kappa}{2} h(x) - \frac{\overline{r}}{h(x)} \right) h(x)\,dx \\ &\quad + \lambda \left(\int_{-\overline{x}}^{\overline{x}} h(x)\,dx - n \right) \\ &= \int_{-\overline{x}}^{\overline{x}} \left\{ \left(\theta\left[2u\left(H(x+2u) - H(x-2u)\right) - \Delta(x)\right] + \lambda - \frac{\kappa}{2} h(x) \right) h(x) - \overline{r} \right\} dx - \lambda n. \end{aligned}$$

The problem is not amendable to optimal control techniques because it is not possible to express the equations of motion as functions of just x, but not functions of $x + 2u$ and $x - 2u$. I proceed by a variational argument adapted from Fujita and Thisse (2002, pp.180–181). First, if there is one additional agent at x, the Lagrangian will increase by the amount,

$$\theta\left[2u\left(H(x+2u) - H(x-2u)\right) - \Delta(x)\right] + \lambda - \kappa h(x).$$

With one additional agent at x, however, there will be an increase in expected gross output for each agent located along the interval $[x - 2u, x + 2u]$ by $\theta 2u$. Each such agent will have to travel to x to trade with the given agent, and on average the trade cost is $\Delta(x)/m$. Thus, the externality generated by the additional agent at x is equal to

$$\left(H\left(x+2u\right)-H\left(x-2u\right)\right)\times\theta\left(2u-\frac{\Delta\left(x\right)}{m}\right).$$

The optimality condition is then

$$2\theta\left[2u\left(H\left(x+2u\right)-H\left(x-2u\right)\right)-\Delta\left(x\right)\right]+\lambda-\kappa h\left(x\right)=0. \tag{67}$$

Differentiating yields (29). The next step is to derive the optimality condition for $\overline{x}$. To this end, differentiate $\mathcal{L}$ with respect to $\overline{x}$ and with symmetry,

$$2\left\{\left(\theta\left[2u\left(H\left(\overline{x}+2u\right)-H\left(\overline{x}-2u\right)\right)-\Delta\left(\overline{x}\right)\right]+\lambda-\frac{\kappa}{2}h\left(\overline{x}\right)\right)h\left(\overline{x}\right)-\overline{r}\right\}.$$

With the increase in $\overline{x}$, there are $2h\left(\overline{x}\right)$ additional agents at the borders. Using a similar argument to the one employed in deriving (67), the externality generated is

$$2h\left(\overline{x}\right)\times\left(H\left(\overline{x}+2u\right)-H\left(\overline{x}-2u\right)\right)\times\theta\left(2u-\Delta\left(\overline{x}\right)/m\right).$$

The optimality condition is thus

$$\left(2\theta\left[2u\left(H\left(\overline{x}+2u\right)-H\left(\overline{x}-2u\right)\right)-\Delta\left(\overline{x}\right)\right]+\lambda-\frac{\kappa}{2}h\left(\overline{x}\right)\right)h\left(\overline{x}\right)-\overline{r}=0.$$

Evaluating (67) at $\overline{x}$ and substituting the resulting expression into the above give (21).

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