**Arrow-Pratt measure of risk aversion**

Let the von-Neumann Morgenstern utility function be $u(w)$. The Arrow-Pratt measure of absolute risk aversion is defined as

$$A_u(w) = -\frac{u''(w)}{u'(w)}$$

**Theorem.** The Arrow-Pratt measure associated with utility function $v$ is larger than that associated with utility function $u$ for all values of $w$ if and only if there exists an increasing and concave function $h$ such that $v(w) = h(u(w))$.

**Proof.** Define $h(x) = v(u^{-1}(x))$. This function is well defined because both $u$ and $v$ are strictly increasing. Also the $h$ function so defined is an increasing function. Let $x = u(w)$, then we can write $h(u(w)) = v(w)$. Differentiate both sides with respect to $w$ to get:

$$h'(u(w))u'(w) = v'(w)$$

Differentiate with respect to $w$ again:

$$h''(u(w))[u'(w)]^2 + h'(u(w))u''(w) = v''(w)$$

Divide the first equation by the second:

$$h''(u(w))u'(w)/(h'(u(w))) - A_u(w) = -A_v(w)$$

Hence $A_v(w) \geq A_u(w)$ if and only if $h'' \leq 0$.

Consider an indifference curve $w_2 = f(w_1)$ in the state-consumption space:

$$p_1u(w_1) + p_2u(f(w_1)) = u_0$$

Differentiate with respect to $w_1$ to get:

$$p_1u'(w_1) + p_2u'(f(w_1))f'(w_1) = 0$$

Differentiate with respect to $w_1$ again to get:

$$p_1u''(w_1) + p_2u''(f(w_1))[f'(w_1)]^2 + p_2u'(f(w_1))f''(w_1) = 0$$
Therefore,
\[
f''(w_1) = -\frac{p_1u''(w_1) + p_2u''(f(w_1))[f'(w_1)]^2}{p_2u'(f(w_1))}
= -\frac{p_1u''(w_1) + p_2u''(f(w_1))[-p_1u'(w_1)/p_2u'(f(w_1))]^2}{p_2u'(f(w_1))}
= \frac{p_1u'(w_1)}{[p_2u'(f(w_1))]^2} \left( p_2u'(f(w_1))A(w_1) + p_1u'(w_1)A(f(w_1)) \right)
\]

Along the certainty line, where \( w_2 = w_1 = w \), we have
\[
f''(w) = \frac{p_1(p_1 + p_2)}{p_2^2} A(w)
\]

So the indifference curve in the state-consumption space is “more curved” if the Arrow-Pratt measure is larger.

If \( A_v(w) \geq A_u(w) \) for all \( w \), then the set of acceptable gambles given utility function \( u \) contains the set of acceptable gambles given utility function \( v \). To see this, let a gamble \( X \) be unacceptable given utility function \( u \). Then
\[
E[u(X)] < u(w)
\]

This is equivalent to
\[
h(E[u(X)]) < h(u(w))
\]

Since \( h \) is concave, the above inequality implies
\[
E[h(u(X))] < h(u(w))
\]

which is the same as
\[
E[v(X)] < v(w)
\]

Hence any gamble which is unacceptable to \( u \) will be unacceptable to \( v \).

The Arrow-Pratt measure is also related to the “certainty-equivalent” of a gamble. Suppose there is a small gamble such that a person’s final wealth is \( w + \varepsilon \), where \( \varepsilon \) is a random variable with mean 0 and variance \( \sigma^2 \). Given risk aversion, forcing him to take the gamble is equivalent to losing \( x \) dollars for sure, where
\[
E[u(w + \varepsilon)] = u(w - x)
\]
Take a second-order Taylor expansion on the left hand side and a first-order expansion on the right hand side:

\[ E[u(w) + \varepsilon u'(w) + (\varepsilon^2/2)u''(w)] \approx u(w) - xu'(w) \]

This equation gives

\[ x \approx (\sigma^2/2)A_u(w) \]

More generally, let \( x_u \) and \( x_v \) be the certainty equivalents given utility functions \( u \) and \( v \) respectively. Suppose \( A_v(w) \geq A_u(w) \) for all \( w \), then \( v(w) = h(u(w)) \) for some increasing and concave \( h \). We have

\[ E[v(X)] = v(w - x_v) \]

which is equivalent to

\[ E[h(u(X))] = h(u(w - x_v)) \]

Jensen’s inequality gives

\[ h(E[u(X)]) > h(u(w - x_v)) \]

That is,

\[ u(w - x_u) = E[u(X)] > u(w - x_v) \]

So, \( x_u < x_v \).