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ORIGINAL ARTICLE

# Macroeconomic Conditions, Firm Characteristics, and Credit Spreads

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**Abstract** We study a structural model that allows us to examine how credit spreads are affected by the interaction betweeen macroeconomic conditions and firm characteristics. Unlike most other structural models, our model explicitly incorporates equilibrium macroeconomic dynamics and models a firm's cash flow as primitive processes. Corporate securities are priced as contingent claims written on cash flows. Default occurs when the firm's cash flow cannot cover the interest payments and the recovery rate is dependent on the economic condition at default. Our model produces the folloBarLinewing predictions: (i) credit spread is mostly negatively correlated with interest rate; (ii) credit spread yield curves are upward sloping for low-grade bonds; (iii) firm characteristics have significant effects on credit spreads and these effects also vary with economic conditions. These predictions are consistent with the available empirical evidence and generate implications for further empirical investigation.

**Keywords** Default risk · Macroeconomic conditions · Credit spreads

**JEL Classifications** G12 · G13 · E43 · E44

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## Introduction

This article presents a theoretical framework for risky debt valuation that explicitly incorporates macroeconomic conditions. We model directly a firm's cash flow process which is embedded in a Lucas-type exchange economy where the growth rate of the aggregate output (i.e., GDP) is mean-reverting. A risky debt written on the firm's cash flow in this economy is then valued using the contingent claim approach. Our approach provides a direct link between market risk and credit risk and enables us to examine macroeconomic, industry- and firm-specific determinants of credit spread levels and changes. Furthermore, our model allows us to assess the cross-sectional differences in the sensitivity of credit spreads to the changes in macroeconomic conditions.

Valuation of risky debt is central to corporate financing choices and credit investors' portfolio management. Many models have been proposed for valuing risky debt following the pioneering work of Black and Scholes (1973) and Merton (1974). This *structural* approach takes as given the dynamics of the risk-free interest rate as well as the dynamics of the asset value of the issuing firm. Default is triggered when the asset value falls below a pre-specified boundary level. Corporate bonds are then valued as contingent claims on the firm's assets.[1] This is the approach we follow in this paper.[2]

Although conceptually elegant, the structural models have had limited success in matching with empirical data. While there is qualitative evidence supporting the Merton-type models (see, for instance, Sarig and Warga, 1989, Titman and Torous 1989, and Bohn 1999), three empirical puzzles remain. First, the magnitude of credit spreads predicted by theoretical models is inconsistent with historical observations. Jones et al. (1984) and, more recently, Huang and Huang (2003) show that credit spreads predicted by the structural models are significantly below the observed levels, especially for high-grade bonds. In assessing empirical performance of several notable structural models, Eom et al. (2004) find that these models to varying degrees tend to underestimate credit spreads for high quality bonds, but overestimate those for junk debt. Those empirical studies indicate that the accuracy of structural models is a major concern.

Second, the predicted shape of the credit yield spread curve for speculative-grade bonds is at odds with historical observations. Helwege and Turner (1999) document that the yield spread curves for high-yield corporate bonds are upward sloping. This finding contradicts the prediction of humped-shape high-yield credit spread curves from most Merton-type structural models, with the notable exception of Collin-

[1] Numerous extensions and generalizations along this line of approach include Black and Cox (1976), Geske (1977), Brennan and Schwartz (1984), Fischer et al. (1989), Kim et al. (1993), Longstaff and Schwartz (1995), Leland (1994, 1998), Leland and Toft (1996), Anderson and Sundaresan (1996), Mella-Baral and Perraudin (1997), Zhou (2001), Duffie and Lando (2001), Collin-Dufresne and Goldstein (2001), Titman et al. (2004).

[2] An alternative approach is the *reduced-form* approach which assumes that default follows some intensity process governed by exogenous random shocks. Papers adopting this approach include Jarrow and Turnbull (1995), Nielsen and Ronn (1996), Jarrow et al. (1997), Lando (1998), Duffie and Singleton (1997, 1999), and Madan and Unal (1998). This approach has the advantage of flexible formulation that allows for a better fit to historical data and can price credit derivatives more easily, but it comes at the expense of lacking economic intuitions as the sources and determinants of default risk in these models are without clear economic interpretations.

Dufresne and Goldstein (2001) in which firms' leverage ratios are exogenously assumed to be mean-reverting.

Third, some fundamental determinants of credit spreads remain elusive. Collin-Dufresne et al. (2001) show that "variables that should in theory determine credit spread changes have rather limited explanatory power." They also demonstrate that the unexplained changes are likely driven by a common factor. Duffie and Singleton (1997) observe that a substantial fraction of swap spreads variation is left unexplained by their model. Elton et al. (2001) illustrate that default risk factors "account for a surprisingly small fraction" of credit spreads. Meanwhile, although it is generally accepted that the correlation between interest rates and credit spreads is negative (see, e.g., Duffee, 1998), Alessandrini (1999) finds evidence for time-series and cross-sectional variations in this correlation.

The collective evidence seems to suggest that these structural models are misspecified as some systematic factors that affect default risk and credit spread dynamics are not properly incorporated. It is well known that interest rates and corporate bond yield spreads fluctuate over business cycles. Aggregate and firm-level outputs critically depend on the state of the economy. Therefore, macroeconomic fundamentals such as economic growth rate should play an important role in determining credit spread dynamics. A growing body of empirical studies has shown that macroeconomic conditions affect yield spreads. Fama and French (1989) find that credit spreads widen when economic conditions are weak. Wilson (1997a,b) and Duffie et al. (2005) find that macroeconomic variables can help explain a significant portion of default rates or yield spread changes. Bakshi et al. (2004) and Elton et al. (2001) show that a substantial portion of corporate bond credit spreads may be explained by factors that we commonly use to model risk premiums for common stocks. Furthermore, Korajczyk and Levy (2003) document that macroeconomic conditions account for 12% to 51% of the time-series variation in firms' leverage between 1984 and 1998, and leverage has been shown to have significant explanatory power for yield spread levels and changes (Collin-Dufresne et al., 2001 and Elton et al., 2001).

A number of authors have recognized the importance of macroeconomic conditions for assessing credit risk and credit spread dynamics. Jarrow and Turnbull (2000) propose that incorporating macroeconomic variables may improve a reduced-form model. Collin-Dufresne et al. (2001) point to "the need for further work on the interaction between market risk and credit risk-that is, general equilibrium models embedding default risk." Moreover, Duffie and Singleton (2003) conjecture that the effects of macroeconomic business cycles on spreads are one of the possible interpretations of the observed negative correlations between credit spreads and yields on Treasury bonds of comparable maturities. Surprisingly, little theoretical work has been done to examine the relation between credit spread dynamics and the state of the economy in an equilibrium setting.

In this paper we explore the effects of macroeconomic conditions on credit spreads from a theoretical perspective. Specifically, we first solve for the equilibrium in a Lucas (1978) exchange economy, in which the growth rate of the economic output is mean-reverting and the representative investor has a constant relative risk aversion (CRRA) utility function. Within this economy, we study a firm whose cash flow growth has both systematic and firm-specific components, with its firm-specific component also following a mean-reverting process. The firm issues a bond with continuous coupons and a finite maturity and default occurs when the firm's cash flow fails to cover the interest payment. Consistent with empirical evidence, our model assumes a stochastic default recovery rate that is dependent on the macroeconomic condition at the moment of default. The risky bond is then valued using the contingent claim approach now standard in the literature [see, e.g., Longstaff and Schwartz (LS, 1995) and Collin-Dufresne and Goldstein (CDG, 2001)].

While many of the structural models do consider one macroeconomic variable, the risk-free interest rate, it is treated as an exogenous variable. In fact, it is endogenously determined by market equilibrium depending on fundamental macroeconomic variables such as GDP growth rates, aggregate market volatility, and investors' risk preferences that also affect credit spread dynamics. Therefore, treating the effect of macroeconomic variables, and particularly the risk-free rate, in a reduced form as done in the existing literature will mask true determinants of credit risk and the dynamics of credit spreads. We explicitly consider the equilibrium of the macroeconomy in which the pricing kernel and the risk-free rate are determined jointly and then used to price the equity and debt of a firm in a consistent manner.[3] Moreover, we model a firm's cash flow instead of its asset value because cash flow characteristics are more easily observable and liquidity constraints are a major cause for default. This approach also allows us to investigate differential sensitivity of asset value to shocks in cash flows and in the discount rate (Campbell and Vuolteenaho, 2004).

We calibrate our model to historical default frequencies and leverage ratios, similar to the approach in Huang and Huang (2003) and Leland (2004). Since our model focuses on default risk, its predicted yield spreads will inevitably deviate from empirical observations because of other known factors impounded in credit spreads such as liquidity and taxes not considered here. Nevertheless, the prediction of credit spreads from our model fares better than other well-known structural models. Our model generates higher yield spreads for high-grade bonds than other models which tend to underestimate the spreads for these bonds, yet for high yield bonds, our model produces smaller yield spreads than other models which are shown to over-predict credit spreads for very risky bonds (Eom et al., 2004). More strikingly, our model generates upward-sloping yield spread curves for speculative-grade bonds, which are consistent with recent empirical evidence and contrary to other structural models, except for the CDG model.

Our comparative static analysis yields results that manifest the significant impact of macroeconomic conditions on credit spread changes. First, credit spreads are counter-cyclical, widening during recessions and narrowing during economic expansions. Because there is a one-to-one relation between economy growth rate and risk-free interest rate, our model demonstrates the economic underpinning for the observed negative correlation between interest rate and credit spreads.

Second, credit spreads increase with volatility of the economic growth rate. This result is intuitively understandable as a firm is more likely to experience cash flow shortfalls in a more volatile economic environment, and hence more likely to default. Credit spreads also widen when investors are more risk averse. It is believed that investors become more risk averse during economic downturns, and this effect has been linked to the "flight to quality" phenomena. Although we do not explicitly

[3] A similar approach has been used in pricing equity options (Bakshi and Chen, 1997) and in studying the effect of labor income on asset prices (Santos and Veronesi, 2006).

model the endogenous change of investors' preferences, these comparative static analysis results provide a gauge of the sensitivity of credit spreads to changes in macroeconomic conditions.

Lastly, our model also has cross-sectional implications for credit spread dynamics and for the effect of the interaction between macroeconomic conditions and industry characteristics. We find that credit spreads decrease with the current firm-specific growth rate and increase with its volatility. The correlation between the firm-level cash flow and the aggregate output, which can be thought of as an industry characteristic, plays a significant role in determining credit spreads and their changes.

A few recent papers are related to our study. Hackbarth et al. (2004) study the effect of aggregate shocks on optimal capital structure choices and credit risk in a model with an exogenous and constant risk-free rate. On the other hand, David (2004) and Marsh and Yan (2002) investigate how learning about the state of the economy can affect default risk of a debt written on the aggregate output. David (2004) also considers the effect of inflation. These papers, however, face the challenge of interpreting the resulting credit spreads associated with the aggregate output and their relevance to empirical observations. Moreover, Pesaran et al. (2003) provide a global perspective of the empirical evidence on macroeconomic dynamics and credit risk.

The rest of the paper is organized as follows. Section 2 introduces our modeling framework. We first lay out the economic structure and derive the equilibrium of the macroeconomy, and then we describe the cash flow process of an individual firm and derive the valuation of its risky debt using the contingent claim approach. Section 3 examines the calibration of the model. The analysis of our model is in Section 4. We offer concluding remarks in Section 5. All technical details are provided in the Appendix.

## The Model

### The Economy

We consider a Lucas (1978)-type exchange economy. In this economy, there could be many firms and multiple risk factors, such as the one studied by Bakshi and Chen (1997). Without loss of generality for our purpose here, we model the aggregate output directly. The total output of this economy, $D(t)$, is described by the following process

$$\frac{dD(t)}{D(t)} = \mu(t)dt + \sigma_D(t)dZ_D(t), \tag{1}$$

where $Z_D(t)$ is a standard Brownian motion. Unless explicitly specified, all processes are under the real probability measure $\mathbb{P}$ in the probability space $\Omega$ with an information filtration $\mathcal{F}_t$, satisfying all regularity conditions. We assume that $D(t)$ has constant volatility, $\sigma_D(t) = \sigma_D$, and its growth rate is mean-reverting,

$$d\mu(t) = \kappa(\overline{\mu} - \mu(t))dt + \sigma_\mu dZ_D(t), \tag{2}$$

where $\kappa$ is the speed of mean reversion, $\overline{\mu}$ the long-run mean, and $\sigma_\mu$ the (constant) growth rate volatility. Therefore the aggregate economy is characterized by a one-factor model.[4] There is a risk-free bond in the economy with zero net supply. The bond price $B(t)$ is defined by the following process,

$$dB(t) = r(t)B(t)dt. \tag{3}$$

The instantaneous risk-free interest rate $r(t)$ is endogenously determined as an equilibrium outcome. The economy specified here is similar to that discussed in Goldstein and Zapatero (1996).

We assume that there is a representative investor who has CRRA utility over consumption $C_t$ with relative risk aversion coefficient $\gamma$ and time discount factor $\delta$,

$$U_t = \mathbf{E}_t\left[\int_0^\infty e^{-\delta s}\frac{C_{t+s}^{1-\gamma}-1}{1-\gamma}ds\right]. \tag{4}$$

The investor chooses the optimal consumption-investment rule to maximize his expected lifetime utility. The first order condition yields the stochastic discount factor (SDF), or pricing kernel, $\pi(t)$ which prices all the assets and payoffs in this economy,

$$\pi(t) = e^{-\delta t}C_t^{-\gamma}. \tag{5}$$

Since in this endowment economy the output is perishable and the total consumption of the entire economy comes from the aggregate output, also known as the dividends, in equilibrium the price of a claim to the aggregate output will be adjusted such that the total consumption equals the dividends, $C_t = D_t$. Using this equilibrium condition and Itô's Lemma, we have

$$\frac{d\pi(t)}{\pi(t)} = -\left(\delta + \gamma\mu(t) - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2\right)dt - \gamma\sigma_D dZ_D(t). \tag{6}$$

Therefore the instantaneous riskfree rate is given by

$$r(t) = -\frac{1}{dt}\mathbf{E}_t\left(\frac{d\pi(t)}{\pi(t)}\right) = \delta + \gamma\mu(t) - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2, \tag{7}$$

and the market price of risk is constant,

$$\theta = \gamma\sigma_D. \tag{8}$$

The following lemmas give the standard results for the prices of risk-free discount bonds and the risky asset.

[4] The growth rate $\mu(t)$ may be unobservable and have its own shocks that are orthogonal to the shocks to $D(t)$. In this case, its filtering process based on the observations of $D(t)$ will make its inferred value follow the process in (2). See Detemple (1986), Dothan and Feldman (1986) and Gennotte (1986) for the separation principle in a partially observable economy.

LEMMA 1 *The time-t price of the risk-free discount bond which pays* 1 *with certainty at maturity time T is given by*

$$P(t,T,r(t)) = e^{A(t,T)-B_\kappa(t,T)r(t)}, \tag{9}$$

*where*

$$A(t,T) = -\left[\bar{r} - \frac{1}{2}\gamma^2\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)\right](T-t) + \bar{r}B_\kappa(t,T)$$
$$-\frac{1}{2}\gamma^2\left[\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{2\sigma_\mu^2}{\kappa^2}\right)B_\kappa(t,T) - \frac{\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t,T)\right],$$

$$B_\kappa(t,T) = \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa}.$$

**Proof:** See Appendix. ■

LEMMA 2 *The present value of the aggregate output $D(t)$ at time t is given by*

$$S(t) = D_t\int_t^\infty \exp(\psi(t,s;r(t)))ds, \tag{10}$$

*where*

$$\psi(t,s;r(t)) = \left[-\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}\bar{r} + \frac{\gamma(1-\gamma)}{2}\sigma_D^2 + \frac{1}{2}(1-\gamma)^2\left(\sigma_D^2 + \frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)\right](s-t)$$
$$+\left[\frac{1-\gamma}{\gamma}(r(t)-\bar{r}) - \frac{1}{2}(1-\gamma)^2\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{2\sigma_\mu^2}{\kappa^2}\right)\right]B_\kappa(t,s)$$
$$+\frac{1}{2}(1-\gamma)^2\frac{\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t,T).$$

**Proof:** See Appendix. ■

The unlevered equity value formula (10) will help to establish the transversality condition for our model parameters. The risk-free discount bond price is affine and the risk-free rate process is the same as the seminal equilibrium model of Vasicek (1977). This economy is completely characterized by the pricing kernel $\pi(t)$. Below we study an individual firm in this economy.

## The Firm

We examine a firm whose cash flow is a tiny portion of the aggregate output. The cash flow the firm generates, $K(t)$, has the following dynamics:

$$\frac{dK(t)}{K(t)} = (\beta\mu(t) + \xi(t))dt + \sigma_K\rho dZ_D(t) + \sigma_K\sqrt{1-\rho^2}dZ_K(t), \tag{11}$$

$$d\xi(t) = \lambda(\bar{\xi} - \xi(t))dt + \sigma_\xi dZ_K(t). \tag{12}$$

where $Z_K(t)$ is a standard Brownian motion independent of $Z_D(t)$, $\rho$ is the correlation between the firm-level cash flow process and the aggregate output process, $\sigma_K$ is the volatility of the firm's cash flow. The drift term, $m(t) = \beta\mu(t) + \xi(t)$, is the current firm-level cash flow growth rate, and $\xi(t)$ is the firm-specific growth rate which is independent of the growth rate of the aggregate output, $\mu(t)$, and assumed to follow a mean-reverting process, with mean-reverting speed $\lambda$, long-run mean $\overline{\xi}$ and volatility $\sigma_\xi$. The sensitivity of firm growth to economic growth,

$$\beta = \mathrm{Cov}\left(\frac{dK(t)}{K(t)}, \frac{dD(t)}{D(t)}\right) \Big/ \mathrm{Var}\left(\frac{dD(t)}{D(t)}\right) = \rho\frac{\sigma_K}{\sigma_D},$$

can be thought of as the cash flow beta in Campbell and Vuolteenaho (2004).

The cash flow $K(t)$ is a part of the aggregate output of the economy, $D(t)$. We can think of $D(t)$ as a forest and $K(t)$ as an individual tree in this forest. We do not explicitly model the rest of the economy. Following the reasoning of Bakshi and Chen (1997), we assume that the rest of the economy $D(t) - K(t)$ follows a process such that $K(t)$ and $D(t) - K(t)$ will aggregate to $D(t)$. Therefore, we can use the pricing kernel of the economy to evaluate an all-equity firm whose cash flow is described by (11) and (12). The following lemma provides such valuation.

LEMMA 3 *The unlevered equity value of the firm with cash flow described by (11) and (12) is given by*

$$S_K(t) = K_t \int_t^\infty \exp(\psi_K(t, s; \mu(t), \xi(t)))ds, \tag{13}$$

*where*

$$\begin{aligned}\psi_K(t, s; \mu(t), \xi(t)) = \Bigg[ & -\delta + (\beta-\gamma)\overline{\mu} + \overline{\xi} + \frac{1}{2}\gamma\sigma_D^2 - \frac{1}{2}\sigma_K^2 + \frac{1}{2}(\rho\sigma_K - \gamma\sigma_D)^2 \\ & + \frac{1}{2}\sigma_K^2(1-\rho^2) + (\beta-\gamma)(\rho\sigma_K - \gamma\sigma_D)\frac{\sigma_\mu}{\kappa} + \sqrt{1-\rho^2}\frac{\sigma_K\sigma_\xi}{\lambda} \\ & + \frac{1}{2}\frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2} + \frac{1}{2}\frac{\sigma_\xi^2}{\lambda^2}\Bigg](s-t) + (\beta-\gamma)(\mu(t) - \overline{\mu})\mathrm{B}_\kappa(t, s) \\ & + (\xi(t) - \overline{\xi})B_\lambda(t, s) - \left[(\beta-\gamma)(\rho\sigma_K - \gamma\sigma_D)\frac{\sigma_\mu}{\kappa} + \frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2}\right]B_\kappa(t, s) \\ & - \left[\sqrt{1-\rho^2}\frac{\sigma_K\sigma_\xi}{\lambda} + \frac{\sigma_\xi^2}{\lambda^2}\right]B_\lambda(t, s) \\ & + \frac{1}{2}\frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t, s) + \frac{1}{2}\frac{\sigma_\xi^2}{\lambda^2}B_{2\lambda}(t, s)\end{aligned}$$

**Proof:** See Appendix. ■

Our model does not explicitly examine the firm's investment opportunities and capital structure decisions. Following the literature, we assume that the Modigliani-Miller theorem holds, which means that the firm's financing decisions do not affect its enterprise value. Therefore, the firm's value will remain the same even if it alters its capital structure in the future. We may extend our model in the dimension along which capital structure matters for the total firm value and the firm chooses optimal capital structure to maximize the firm value.[5] This, however, will introduce more complexity before we have a better understanding of the pricing effect of macroeconomic conditions. Therefore, as an initial step towards a more comprehensive study, we maintain our current setting and apply the contingent claim approach to value a bond that uses the cash flow $K(t)$ as collateral.

Most other structural credit risk models in the literature start from the asset value process. However, an exogenously assumed asset value process may not be internally consistent with a pricing kernel that prices securities in a unified framework. In those models, default occurs when the firm value falls below some threshold. In reality this is rarely the case. Firm value falling below a threshold may precipitate some interest payments, but the fundamental reason for default is that the firm does not have enough cash for its interest payments.[6] Uhrig-Homburg (2005) explicitly models cash flow shortage as an endogenous bankruptcy reason in the presence of equity-issuance costs.[7]

Here we model the firm's cash flow as a primary process, and the firm defaults when it does not have enough cash to pay its dues. Other studies have also examined cash flow rather than firm value process. Kim et al. (1993) argue that firm defaults when its cash flow is not sufficient for its coupon payments, but they specify the cash flow in such a way that the firm's asset value follows a geometric Brownian motion and still treat the firm value as the fundamental variable. Goldstein et al. (2001) model the firm-level cash flow directly in determining optimal dynamic capital structure choice. Titman et al. (2004) consider endogenous cash flow processes with investment decisions and investigate the effect of investment flexibility on credit spreads. In addition, Marsh and Yan (2002) study credit spread dynamics for defaultable debt on the aggregate output.

## Bond Valuation

Given the firm's cash flow process, we can value any contingent claims written on this cash flow. We assume that the firm has a bond at time 0 and focus on the pricing of such a contingent claim.

The risky debt has a face value $F$, coupon payment rate $c$, and maturity $T$. This risky bond is pledged on the firm cash flow $K(t)$. During each period, $\Delta t$, the firm will pay the bondholders a fixed coupon, $c\Delta t$, before the bond matures. The firm defaults when its cash flow is not enough to cover the coupon payment, $K < c$. In that event, either reorganization or liquidation is imposed and the bondholders

[5] See, among others, Fischer et al. (1989), Leland (1994, 1998), Leland and Toft (1996), Goldstein et al. (2001), and Titman et al. (2004). Hackbarth et al. (2004) examine the effect of macroeconomic shocks to optimal capital structure decisions and credit spreads, but they use a partial equilibrium framework in which the risk-free rate is assumed to be constant.

[6] See Wruck (1990) for the information problem in financial distress and the difference between cash flow triggered default and asset value triggered default.

[7] Here we exclude the possibility of strategic default because the firm value will never fall below zero. For studies on strategic debt services, see Anderson and Sundaresan (1996, 2000), Mella-Baral and Perraudin (1997), Mella-Baral (1999), Fan and Sundaresan (2000), and John et al. (2003).

recover a fraction $w(\cdot)$ of the face value $F$. This recovery of face value at default (RFV) assumption is shown to be most consistent with empirical evidence among alternative default recovery assumptions (Guha 2003). The payoff stream of this defaultable bond is

$$g(t) = c \cdot 1(t \leq T) \cdot 1(t < \tau) + F \cdot \delta(t - T) \cdot 1(t < \tau) + \omega(\cdot) F \cdot \delta(t - \tau) \cdot 1(t \leq T), \quad (14)$$

where $\tau = \inf\{t : K(t) < c\}$ is the first passage time which represents the time of default, and $\delta(t - \tau)$ is the Kronecker delta.

Extant studies on bond valuation in the literature assume constant default recovery rate. Beyond tractability reasons, this assumption is not supported by empirical evidence. For example, in a comprehensive investigation of all defaulted bonds, Altman and Kishore (1996) find that recovery rates are time varying. Further, Collin-Dufresne et al. (2001) argue that "even if the probability of default remains constant for a firm, changes in credit spreads can occur due to changes in the expected recovery rate. The expected recovery rate in turn should be a function of the overall business climate."

Here we assume that the recovery rate $\omega(\mu_t)$ depends on the current growth rate of the economy, which is consistent with recent empirical findings of Thorburn (2000), Gupton and Stein (2002), Altman et al. (2002), and Acharya et al. (2003), who collectively show that macroeconomic and industry conditions at the time of default are important and robust determinants of the recovery rate. The intuition for this is also found in Shleifer and Vishny (1992) who show, in an industry equilibrium setting, that a firm's liquidation value is lower when its competitors are experiencing cash flow problems. We capture this relation in a parsimonious way by assuming

$$w(\mu_t) = a + b\mu_t, \quad (15)$$

where $b \geq 0$. Although such a linear specification for the recovery rate would allow it to go beyond the $[0, 1]$ range, in our model the volatility of $\mu_t$, $\sigma_\mu$, is so low that the actual recovery rate in our investigation should stay in the $[0, 1]$ range.

Moreover, Acharya et al. (2003) show that the factors affecting default risk are only weakly dependent on the factors affecting recovery rate. Therefore, for analytical tractability, we assume that default risk and recovery rate risk are independent, that is,

$$\mathrm{E}[w(\mu_\mathrm{t})\delta(t - \tau)] = E[w(\mu_t)]\mathrm{E}[\delta(t - \tau)].$$

Note that since the empirical correlation is marginally positive, our simplification will decrease expected loss, thereby bias downward credit spreads in our calculation.

With these assumptions, we arrive at the following Proposition.

PROPOSITION 1 *The value of the risky debt at $t = 0$ is given by*

$$DV = \mathrm{E}^{\mathbb{Q}}\left[\int_0^T e^{-\int_0^t r(s)ds} g(t)dt\right] = FV - EL, \quad (16)$$

*where*

$$FV = c\int_0^T P(0, t, r(0))dt + F \cdot P(0, T, r(0)) \quad (17)$$

*is the value of a default risk-free bond with an identical payment structure, and*

$$
\begin{aligned}
EL = c\int_0^T P(0,t,r(0))\Gamma(t)dt + (1-a-b\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)])F\cdot P(0,T,r(0))\cdot\Gamma(T)\\
+F\int_0^T\left(P_t(0,t,r(0))(a+b\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)]) + P(0,t,r(0))b\frac{\partial\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)]}{\partial t}\right)\Gamma(t)dt
\end{aligned}
\tag{18}
$$

*is the expected loss of the risky bond, where* $\Gamma(t)\equiv\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\mathbf{1}(\tau\le t)]$ *is the cumulative distribution function of* $\tau$ *in the* $T$*-forward risk neutral measure, which represents the probability that default occurs before time* $t$*, and its density function is* $f(t)\equiv\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\delta(t-\tau)]$, $P_t(0,t,r(0))=\partial P(0,t,r(0))/\partial t$, *the expressions for* $\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)]$ *and* $\frac{\partial\mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)]}{\partial t}$ *are given in the Appendix.*

**Proof:** See Appendix. ∎

The valuation formula for the risky bond is very intuitive. The expected loss given default (LGD) consists of three components: the present value of the sum of all remaining coupon payments, the present value of the loss on the principal, and the present value of the reinvestment on the recovered principal. The yield to maturity of this risky bond $Y$ is implicitly defined by

$$DV=\frac{c}{Y}+\left(F-\frac{c}{Y}\right)e^{-YT}. \tag{19}$$

Similarly, the yield to maturity of a risk-free bond with the same payment structure, $R$, is given by

$$FV=\frac{c}{R}+\left(F-\frac{c}{R}\right)e^{-RT}. \tag{20}$$

Following the extant literature such as LS and CDG, the credit yield spread is defined as $Y-R$.

A key ingredient of the bond valuation formula is the default probability $\Gamma(t)$. In our model, conditioning on the current growth rate, the firm's cash flow is normally distributed and has a two-factor structure. Therefore we can directly apply the flows of probability approach to find the probability distribution function for $\tau$. This approach was first used by LS and subsequently modified by CGD and Huang and Huang (2003). We adapt the formulation of the cumulative default probability from Huang and Huang (2003) as stated in the following Lemma.

Lemma 4 *The probability of default before time t under the risk-neutral forward measure is given by:*

$$\Gamma(t)=\lim_{n\to\infty}\sum_{i=1}^{n}q(t_{i-\frac{1}{2}}),\quad i=1,2,\ldots,n, \tag{21}$$

*where*

$$
\begin{aligned}
t_i &= i\frac{T}{n},\\
q(t_{i-\frac{1}{2}}) &= \frac{N(a(t_i)) - \sum_{j=1}^{i-1} q(t_{j-\frac{1}{2}})N(b(t_i;t_{j-\frac{1}{2}}))}{N(b(t_i;t_{i-\frac{1}{2}}))},\\
a(t_i) &= -\frac{M(t_i,T|X_0,m_0)}{\sqrt{W(t_i|X_0,m_0)}},\\
\mathrm{b}(t_i;t_j) &= -\frac{M(t_i,T|X_{t_j})}{\sqrt{W(t_i|X_{t_j})}},
\end{aligned}
$$

*where*

$$
\begin{aligned}
M(t_i,T|X_0,m_0) &\equiv \mathbf{E}_0^{\mathbb{F}_\mathbf{T}}[X_{t_i}],\\
W(t_i|X_0,m_0) &\equiv \mathbf{Var}_0^{\mathbb{F}_\mathbf{T}}[X_{t_i}],\\
M(t_i,T|X_{t_j}) &= M(t_i,T|X_0,m_0) - M(t_j,T]|X_0,m_0)\frac{\mathbf{Cov}_0^{\mathbb{F}_\mathbf{T}}[X_{t_i},X_{t_j}]}{W(t_j|X_0,m_0)},\\
W(t_i|X_{t_j}) &= W(t_i|X_0,m_0) - \frac{\left(\mathbf{Cov}_0^{\mathbb{F}_\mathbf{T}}[X_{t_i},X_{t_j}]\right)^2}{W(t_j|X_0,m_0)},
\end{aligned}
$$

*where* $X_0 = \log(K_0/c)$ *is current firm coverage ratio,* $m_0 = \beta\mu_0 + \xi_0$ *is current firm growth rate. The expressions for* $E_0^{\mathbb{F}_\mathbf{T}}[X_{t_i}]$, $Var_0^{\mathbb{F}_\mathbf{T}}[X_{t_i}]$, *and* $\mathbf{Cov}_0^{\mathbb{F}_\mathbf{T}}[X_{t_i},X_{t_j}]$ *are given in the Appendix.*

**Proof:** See Appendix. ■

The default probability formula is completely characterized by the conditional and unconditional first two moments of the current firm coverage ratio $X_0$ (i.e., how well the firm's cash flow can cover its interest payments). The series in its expression converges very rapidly. The default probability has the same form under the real measure, with only the expressions for the first moment being modified, as given in the Appendix.

## Calibration of the Model

Similar to Huang and Huang (2003), we choose the parameters by jointly matching the historical leverage ratios and realized default frequencies. However, we do not strictly follow the procedure in Huang and Huang (2003). After fixing other free parameters for various models, Huang and Huang (2003) choose initial firm value, asset risk premium, asset volatility, and recovery rate to match initial leverage ratio, equity premium, cumulative default probability, and recovery rate. Therefore, for each credit rating over a given horizon, the calibration yields a distinctive value of asset volatility. This approach may be inherently inconsistent, as for the same firm, there can be different asset volatilities calibrated to bonds of different horizons.

**Table 1** Baseline parameters used in the calibration. This table summarizes the baseline parameters used in our calibration exercise and comparative static analysis

| Parameter | Symbol | Value |
|---|---|---|
| Preference: | | |
| Risk aversion | $\gamma$ | 2 |
| Time discount factor | $\delta$ | 0.05 |
| Macroeconomic condition: | | |
| Standard deviation of economy growth | $\sigma_D$ | 10% |
| Mean reversion of economy growth rate | $\kappa$ | 1.5 |
| Long-run mean of economy growth rate | $\overline{\mu}$ | 3% |
| Long-run mean of risk-free rate | $\overline{r} = \delta + \gamma\overline{\mu} - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2$ | 8% |
| Standard deviation of growth rate | $\sigma_\mu$ | 2% |
| Current economy growth rate | $\mu_0$ | 4% |
| Firm characteristics: | | |
| Standard deviation of firm cash flow growth | $\sigma_K$ | 12% |
| Correlation between firm and economy | $\rho$ | 0.6 |
| Cash-flow beta | $\beta = \rho\sigma_K/\sigma_D$ | 0.72 |
| Mean reversion of firm-specific growth rate | $\lambda$ | 1 |
| Long-run mean of firm-specific growth rate | $\overline{\xi}$ | 2% |
| Standard deviation of firm-specific growth rate | $\sigma_\xi$ | 2% |
| Current cash flow | $K_0$ | 16 |
| Current firm-specific growth rate | $\xi_0$ | 2% |
| Current firm growth rate | $m_0 = \beta\mu_0 + \xi_0$ | 4.88% |
| Bond feature: | | |
| Face value | $F$ | 100 |
| Coupon rate | $c$ | 8% |
| Recovery rate constant | $a$ | 42.5% |
| Recovery rate amplifier | $b$ | 2.5 |
| Recovery rate mean | $\overline{w} = a + b\overline{\mu}$ | 50% |

Instead, we follow the approach in Leland (2004) by choosing initial cash flow, initial firm-specific growth rate, and volatility of firm growth rate to jointly match leverage ratio and default probabilities at one, four and ten year horizons. While we will not exactly match all default probabilities due to the over-identification problem (four constraints with only three free parameters), we choose those parameters to minimize the difference between predicted and historical default probabilities.

The resulting parameter values are reported in Table 1. The calibration of macroeconomic variables follows the existing evidence in the literature as summarized in Huang and Huang (2003). The risk aversion ($\gamma$) is taken to be 2, which is the typical value used in the literature (e.g., Campbell and Cochrane, 1999).[8] We choose the time discount factor ($\delta$) to be 0.05, which in the range of values used in the literature, and the average growth rate of economic output ($\overline{\mu}$) to be 3% per annum. The volatility of the economic output is set at 10% per annum. This choice of variables allows us to set the long-run mean of the risk-free rate at 8% per annum as indicated in Huang and Huang (2003). Because of the power utility for the representative agent in our model, the equity premium may

[8] Chen et al. (2005) use instead $\gamma = 4$. To avoid an unrealistically high volatility for the risk-free rate in our model, we stick with $\gamma = 2$.

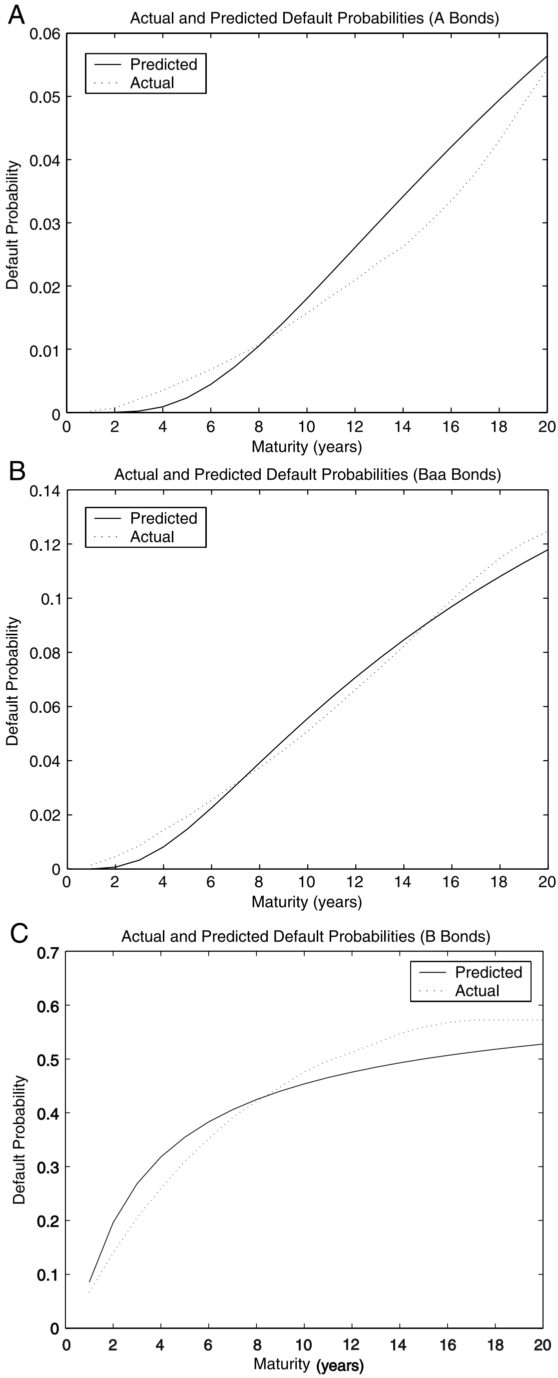


**Fig. 1** Term structure of cumulative default probability. Three panels, from top to bottom, are for A-rated, Baa-rated, and B-rated bonds, respectively. The historical default frequencies are from Moody's Investors Service (2002). The parameters used are provided in Table 1

be lower than the historical average, but it is roughly on par with ex ante estimates (e.g., Claus and Thomas, 2001). In addition, we set the mean-reversion parameter, $\kappa$, to be 1.5, on par with the value used in Longstaff and Schwartz (1995).

On the firm level, there is a wide range of values we could choose for various parameters, depending on the firm characteristics we want to focus on. In our calibration, we define leverage ratio as book debt divided by the market value of the firm,

$$\text{Leverage Ratio} = \frac{F}{S_K(0)}.$$

We obtain different firm leverage ratios by changing current earnings $K_0$ and, consequently, the current coverage ratio $X_0 = \log(K_0/c)$. Although many of the other parameters will be varied in our sensitivity analysis, as a benchmark case, we set the standard deviation of the firm's cash flow at 12% per annum, the correlation between the firm's cash flow and macro-economic output at 0.6, resulting the firm's cash flow beta to be 0.72. The long-run mean of firm-specific growth rate is set at 2% per annum, with a 2% standard deviation and a mean-reversion parameter of 1. For bonds issued by the firm, we use a 8% coupon rate and assume an average recovery rate at 50%, on par with the number used in Huang and Huang (2003). With the stochastic recovery with the same mean, we set $a = 42.5$, and $b = 2.5$.

For this set of parameters, a firm with a leverage ratio 30.58% (with an approximate A rating) has a predicted default probability of 1% if its bond maturity is four years, and 6.39% if its bond maturity is ten years. The historical data from Standard & Poor's and Moody's, as documented by Huang and Huang (2003), show that for A-rated bonds with an average leverage ratio of 31.98%, the corresponding default frequencies are 0.35% and 1.55%, respectively. To further demonstrate qualitatively the fit of our predicted default probability curve with respect to the historical experience, we plot in Fig. 1 three cumulative default probabilities for three credit rating classes, following the approach in Leland (2004).[9]

In our model, both macroeconomic and firm-specific variables jointly determine yield spreads and yield spread changes. The macroeconomic condition variables we will focus on are risk aversion, current growth rate, and volatility of economic growth. Firm-specific variables include current leverage ratio, volatility of the growth rate of cash flow, current firm-specific growth rate, and correlation between economy-wide and firm-level growth which links market risk and credit risk. In addition, the default recovery rate is dependent on the current economic conditions. In the following we use the calibrated parameters to generate predictions for credit yield spreads and carry out a series of comparative static analysis.

## Analysis of Results

In this section, we first discuss the level and the shape of credit yield spread curves generated by our model using the calibrated parameters. We then conduct a comparative static analysis to gauge the effects of macroeconomic conditions, firm characteristics, and the interaction between market risk and credit risk on credit

[9] Duffie et al. (2005) find that incorporating macro-economic variables helps improve corporate default predictions. Due to data limitation, we are not able to make comparisons between their approach and our model in this paper.

yield spread changes. We note that our analysis is done to assess qualitative properties of credit spreads, in the same spirit of Huang and Huang (2003), instead of a more quantitative prediction on a day-to-day basis.

### Level and Shape of Yield Spread Curves

Our model compares favorably to some well-known structural models such as LS and CDG, in predicting the magnitude of credit yield spreads for both investment-grade and speculative grade corporate bonds. Table 2 reports predicted credit spreads from our model for three credit rating classes (A, Baa and B) and two horizons (4 and 10 years). It also lists comparable numbers from other models, including the Longstaff and Schwartz (1995) model (LS), the Leland and Toft (1996) model (LT) and the Collin-Dufresne and Goldstein (2005) model (CDG), based on the estimates in Huang and Huang (2003). The table shows that for high-grade bonds (with A and Baa ratings), our predicted values are substantially larger than other models. While the predicted values are still significantly below the historical averages as expected, due to tax and liquidity issues addressed in the literature, the results indicate our predicted values are about 50% higher for four-year bonds, and more than double in the yield spread for 10-year bonds.

More notable is the fact that for lower grade bonds, such as ones with a B rating, our predicted significantly lower spreads than other models. This is, in our view, an advantage, because tax and liquidity issues should be of concern for high grade bonds as much as, if not more, for low grade bonds. Other models tend to predict much higher credit spreads for low grade bonds which would result in a relatively smaller portion of the spread attributable to these non-default related components. Eom et al. (2004) discuss this concern of over-fitting for low grade bonds by other models. Our predicted values for low grade bonds seem more reasonable and consistent in light of this concern. The reason our model seems to perform better than these other models is because, in our model, there are two distinct uncertainties in the growth rate which follow mean-reverting processes. The combined effect of two uncertainties increases yields for safer bonds; at the same time it drives down the yield spreads for very risky bonds.

Just as important, our model obtains a qualitatively accurate prediction for the shape of the yield spread curve. Our model predicts upward-sloping yield spread curves for speculative grade bonds, as shown in Fig. 2, which is consistent with

**Table 2** Comparison of model predictions of credit spreads. This table reports predicted credit spreads from our model using calibrated parameters and other models from Huang and Huang (2003). LS refers to the Longstaff and Schwartz (1995) model with stochastic interest rate. LT refers to the Leland and Toft (1996) model. CDG refers to the Collin-Dufresne and Goldstein (2005) model

| Horizon and rating | Historical | Our Model | LS | LT | CDG |
|---|---|---|---|---|---|
| 4-year A | 96 | 13.2 | 7.5 | – | 9.9 |
| 4-year Baa | 158 | 56.6 | 25.4 | – | 31.1 |
| 4-year B | 470 | 240.5 | 406.0 | – | 435.3 |
| 10-year A | 123 | 48.1 | 14.5 | 38.5 | 22.5 |
| 10-year Baa | 194 | 109.6 | 38.6 | 59.5 | 52.3 |
| 10-year B | 470 | 318.5 | 341.9 | 408.4 | 371.6 |

recent empirical evidence documented by Helwege and Turner (1999). The finding of Helwege and Turner is considered to be one of the most serious challenges to Merton-type bond pricing models because most of other structural models, with the notable exception of CDG, generate hump-shaped or downward-sloping yield spread curves for junk bonds. In Merton-type models, the drift term of the asset value process is always positive, therefore conditioning on that the firm does not default in the short run, the long-run survival probability for the firm is very high. This is not the case in our model, the growth rate can go negative due to the two mean-reverting components therefore the default probability always increases with maturity. Incidentally, in CDG the default probability increases with maturity because the firm maintains a stationary leverage ratio.

## Macroeconomic Conditions and Yield Spreads

### *Current Aggregate Growth Rate*

Our model predicts that credit spreads widen during economic downturns. Figure 3 shows the yield spread curves for three different values of current economic growth rate $\mu_0 = -1\%, 4\%, 7\%$. Yield spreads decrease significantly when the current growth rate moves from $-1\%$ to $7\%$. For example, for a ten-year maturity Baa bond, yield spread declines to $48.7 - 3$ basis points from $57.2$ basis points as current economic growth rate increases from $-1\%$ to $7\%$. This finding is qualitatively consistent with empirical evidence documented by Fama and French (1989) who show that credit yield spreads display a business cycle pattern: yield spread narrows when economic conditions are strong and widens when conditions are weak. Similar empirical evidence is also found in a number of other studies.

Because, *ceteris paribus*, there is a one-to-one relation between economic growth rate and real risk-free interest rate, as in

$$r(t) = \delta + \gamma\mu(t) - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2,$$

yield spreads should be negatively correlated with the risk-free rate. This prediction is consistent with the finding in Duffee (1998) and confirms the conjecture by Duffie

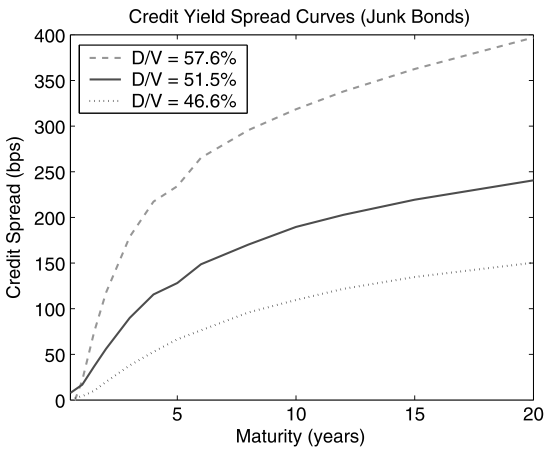


**Fig. 2** Speculative-grade credit spreads. The parameters used are provided in Table 1. Three leverage ratios are considered, corresponding to rating classes around “Ba” and “B”

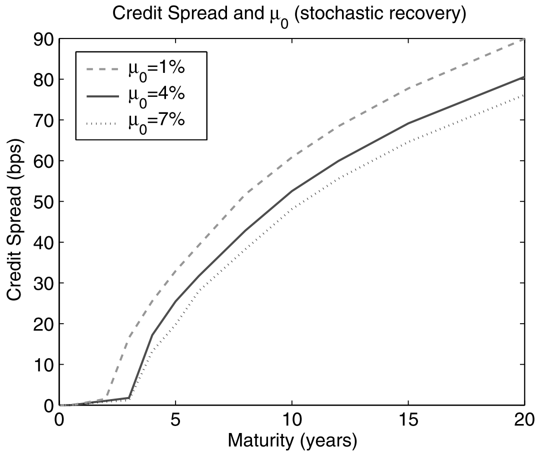


**Fig. 3** Credit spreads for different values of the current economy growth rate $\mu_0$. The parameters used are provided in Table 1

and Singleton (2003) that this negative correlation could be due to macroeconomic factors. The intuition for the negative correlation is as follows. The drift of the firm's cash flow process is positively related to economy growth rate (with a positive cash flow beta). An increase in economic growth rate will increase the drift, and therefore decrease the default probability and credit spreads.

Note that this negative correlation was generated by the positive correlation between firm cash flow growth and economic growth. This is the case in our calibrated parameters because an average firm, which is a part of the aggregate economy, is positively correlated with the economy. Since some firms may be counter-cyclical (negative beta firms), a positive correlation between interest rate and credit spreads can arise for these firms. Because most empirical tests use portfolios and the aggregate cash flow beta in a portfolio beta tend to be positive, as most firms are more pro-cyclical than counter-cyclical, so measured correlations will be negative. But the magnitude of this correlation may not be very large. While Longstaff and Schwartz (1995) find a significantly negative correlation, Duffee (1998) and Jacoby (2002) show that optionality in the callable bonds used in that analysis may account for a significant amount of the negative correlation they found.[10] Duffee (1998) uses straight bonds and finds negative correlations at a much smaller level. The relatively small changes in credit spreads in Fig. 3 in response to the large change in the economic growth rate seems consistent with that result.

*Volatility of Aggregate Economic Growth*

Firms are more likely to default in a volatile market because there is a greater chance for a firm to experience dramatically negative growth and therefore realize a low level of cash flow. Figure 4 graphs the term structure of credit spreads for three different values of aggregate growth volatility: $\sigma_D = 9\%, 11\%, 13\%$. As expected,

[10] Because when the market interest rate increases, the value of the call options will decrease, consequently the price of the bond will increase resulting in a drop in yields and yield spreads.

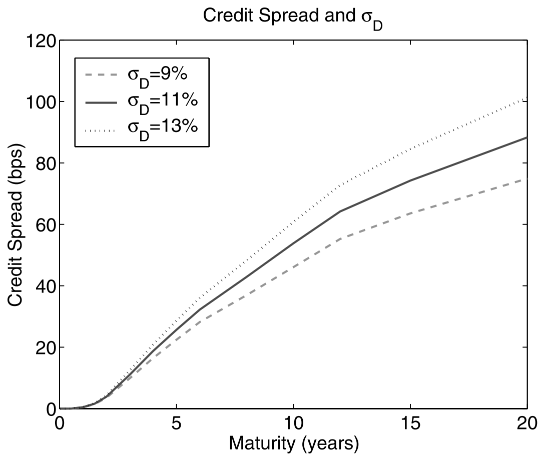


**Fig. 4** Credit spreads for different values of the standard deviation of economy growth $\sigma_D$. The parameters used are provided in Table 1

yield spreads monotonically increase with $\sigma_D$. Yield spreads are substantially larger during volatile markets. As $\sigma_D$ increases from 9% to 13%, yield spread widens by 14.7 basis points (from 46.1 to 60.8 basis points) for ten-year bonds and 26.4 basis points (from 74.9 to 101.3 basis points) for twenty-year bonds. Note this effect is stronger for longer term bonds.

There are two reinforcing effects contributing to the sensitivity of yield spreads to aggregate growth volatility. On one hand, a higher $\sigma_D$ leads to a lower risk-free interest rate $r_t$, due to the precautionary savings motive. Hence, the price of a risk-free bond will be higher and its yield will be lower. On the other hand, the market risk premium $\theta = \gamma\sigma_D$ increases with $\sigma_D$, a higher risk premium will push down the price of the risky bond resulting in a higher yield. These two effects reinforce each other, and subsequently the yield spreads rise with $\sigma_D$. In volatile markets, investors tend to prefer safe securities. Therefore this effect may be associated with the "flight-to-quality" phenomenon.

*Risk Aversion*

Some have argued that changing economic conditions can be characterized by changes in investor's risk aversion, and investors are more risk averse during economic recessions (see, e.g., Campbell and Cochrane, 1999). We plot yield spread curves for three different values of investor risk aversion parameter $\gamma = 0.5, 2, 3.5$ in Fig. 5. Yield spreads increase significantly with risk aversion. The mechanism for $\gamma$ to influence yield spreads is similar to the mechanism for $\sigma_D$, and this effect is also more pronounced for longer maturity bonds. Yield spreads increase by 40 basis points (from 32.9 to 72.9 basis points) for ten-year bonds and 69 basis points (from 52.5 to 121.5 basis points) for twenty-year bonds. Moreover, there is convexity in this relationship: The same amount of change in risk aversion affects credit spreads more at a higher level of risk aversion. When investors are more risk averse, they invest in safer securities; yield spreads will have to widen more to attract investors. This relation between yield spreads and risk aversion is also consistent with the "flight-to-quality" phenomenon.

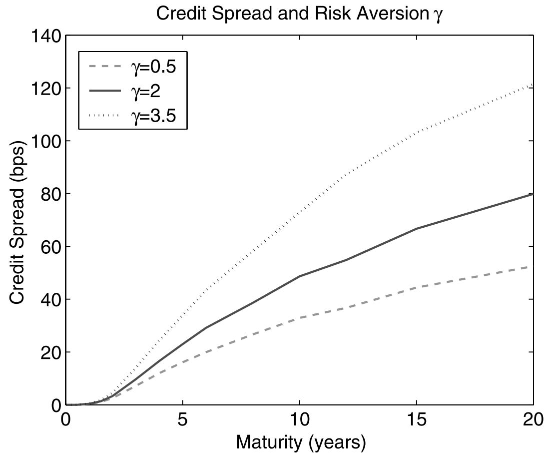


**Fig. 5** Credit Spread and Risk Aversion. The parameters used are provided in Table 1

## Firm Characteristics and Yield Spreads

### *Cash Flow Volatility*

For an individual firm, the more volatile its cash flow is, the more likely it will encounter a shortfall in covering the interest payment, and hence is more likely to default. This reasoning indicates that the cost of capital, or credit spread in this context, should increase with the cash-flow volatility. As shown in Fig. 6, our model predicts that credit spreads rise dramatically (from 25.7 basis points to 72.2 basis points) for ten-year maturity when a firm's cash-flow volatility increases from 10% to 14%. This finding is empirically supported by Minton and Schrand (1999).

Comparing Figs. 4 and 6, we observe that cash flow volatility seems to have a larger impact on credit spreads than the volatility of aggregate output. To the extent that cash flow volatility relates to firm-specific risk, this is consistent with the empirical evidence provided by Campbell and Taksler (2003) who show that the increase in corporate bond yields in the late 1990s can be largely explained by the increase in idiosyncratic firm-level volatility and the explanatory power of idiosyncratic volatility is as good as credit ratings. In addition, empirical evidence has also shown that a firm's total equity volatility significantly influences yield spreads (Collin-Dufresne et al., 2001 and Cremers et al., 2004).

### *Current Firm-Specific Growth Rate*

In our model, firm-level growth rate consists of two components: the market or systematic component and the firm-specific or idiosyncratic component:

$$m_t = \beta\mu_t + \xi_t.$$

It is therefore clear that firm-specific growth rate will have similar effects on credit yield spreads as economic growth rate: yield spreads decrease as growth rate increases. This observation is confirmed by Fig. 7. The difference in yield spreads between two firms having the same cash flow beta, one with 2% firm-specific growth

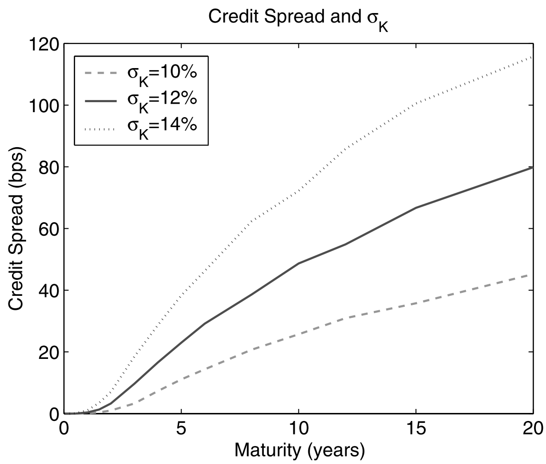


**Fig. 6** Credit spreads for different values of the standard deviation of firm growth $\sigma_K$. The parameters used are provided in Table 1

rate and another with $-2\%$, is 11 basis points (between 59.7 and 48.7 basis points) for ten-year bonds. While we are not aware of any empirical studies examining this issue, our model provides a refutable empirical prediction.

*Cash Flow Beta*

One innovative aspect of our model is that we incorporate a beta specification in a firm's cash flow growth. Therefore the correlation between the growth rate of the firm and the growth rate of the economy, $\rho$, or equivalently, the firm's cash flow beta, plays an important role in determining credit yield spreads. This correlation may be used to characterize different industries. For example, the utility industry may be modestly correlated with aggregate economy, while the financial sector is more correlated with the aggregate economy than the consumer product sector.

Figure 8 graphs the term structure of yield spreads for different values of $\rho$ in different states of the economy. The results are consistent with empirical evidence

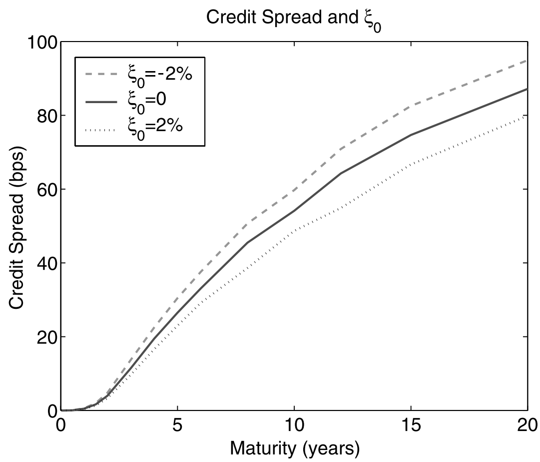


**Fig. 7** Credit spreads for different values of the current firm-specific growth rate $\xi_0$. The parameters used are provided in Table 1

that yield spreads for investment-grade bonds vary across sectors, albeit the magnitude of this effect may be modest. Moreover, the relation between credit spreads and firm cash flow beta depends on the current state of the economy, as shown in the figure. During economic downturns, firms highly correlated with the economy will more likely experience low growth than less correlated firms. Consequently yield spreads will be larger for more economy-sensitive firms. The opposite is true during economic expansions. These predictions provide good refutable hypotheses for empirical tests.

## Concluding Remarks

We have theoretically explored the effects of macroeconomic conditions on credit spread dynamics in an equilibrium framework. Our modeling of fundamental processes (cash flow or EBIT) ensures the internal consistency of our valuation system. Our model is easy to implement and requires very simple inputs: aggregate

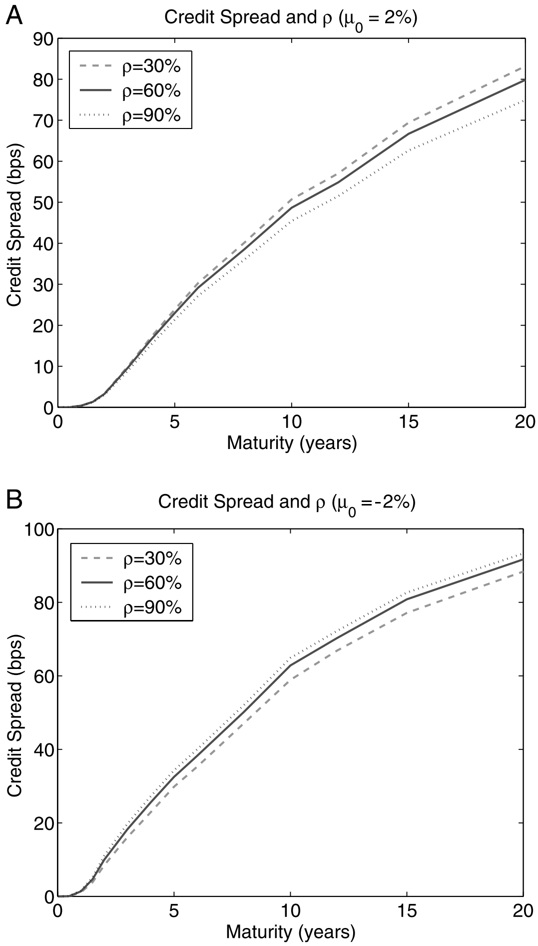


**Fig. 8** Credit spreads for different values of correlation between the economy and firm growth $\rho$. The top panel depicts an economic expansion, and the bottom panel describes an economic downturn. The parameters used are provided in Table 1

economic output (i.e., GDP) and firm-level cash flow variables. We contribute to the risky debt valuation literature by exploring the important link between market risk and credit risk.

We calibrate the model by jointly matching historical leverage ratios and default frequencies, similar to the approach proposed by Huang and Huang (2003) and Leland (2004). Our model compares favorably with other well-known structural defaultable bond pricing models in matching the magnitude of credit spreads for both investment-grade and speculative-grade bonds. Our model also generates upward-sloping yield spread curves for speculative-grade bonds, an empirical feature that most other structural models fail to explain. The success of our model is attributable to the intrinsic link between shocks to aggregate output and firm-level cash flows that we explicitly model in this paper.

Through comparative static analysis, we show that macroeconomic variables are important in explaining a substantial portion of yield spread changes. Industry and firm-level characteristics determine the cross-sectional differences in the level of credit spreads as well as the response of yield spreads to the change in macroeconomic conditions. Our analysis yields results that are either consistent with existing evidence or amenable to further empirical tests.

Our beta specification for firm-level cash flow growth makes our model feasible for credit portfolio risk management. Similar to the portfolio theory in investment, this approach allows us to value each bond individually and then combining them with all correlations taken care of by cash flow betas. Therefore, we may deal with the issue involving correlated defaults.[11] Further exploration of this issue is left for future research.

It is intriguing that our model yields significant improvements in terms of predicting the level of credit spreads over the existing structural models. This advantage comes from two potential sources. One is the correlation between aggregate output and firm-level cash flow. This additional systematic exposure that we consider adds to the risk premium contained in credit spreads. Another source may be due to our assumption that firms will default when their cash flow is not enough to service their debt. This liquidity-induce default precludes the possibility of issuing new equity to deal with cash flow shortage and may be more applicable to firms facing high equity-issuance costs. However, as pointed out by Uhrig-Homburg (2005), such a constraint may lead to a different choice of capital structure and therefore affect credit spreads through additional channels. This suggests some interesting cross-sectional studies to better understand the implication of cash flow constraints for bond pricing and represents a promising venue for further investigation.

Finally, while our study contributes to the literature by modeling credit risk and market risk jointly in an equilibrium framework, it is a first step towards a comprehensive understanding of the issue. Alternative equilibrium frameworks have been explored in Alvarez and Jermann (2000) and Chang and Sundaresan (2005). However, it is difficult to provide quantitative characterization of credit dynamics in these frameworks. Recent work by Chen et al. (2005) is another attempt in this

[11] Our focus on macroeconomic conditions also grants us a unique advantage in reconciling related empirical anomalies. For example, Covitz and Downing (2002) find that "the correlation between many firms' short-term and long-term yield spreads are negative, typically during periods characterized by credit market disruptions." While we do not yet explore this point in our current study, our model clearly has the potential to examine this anomaly.

direction. More effort is needed to have a truly integrated approach that both endogenizes default risk and characterizes credit spread dynamics in an equilibrium.

**Acknowledgments** We would like to thank Lorenzo Garlappi, Gerald Lumer, Weidong Tian, Sheridan Titman and an anonymous referee, as well as participants of the 2004 C.R.E.D.I.T. Conference in Venice, Italy, the 2004 FMA Annual Meeting in New Orleans, and at seminars in Fudan University (Shanghai), Moody's KMV and the University of Texas at Austin for helpful suggestions and comments. This study was completed while Tang was a doctoral student at the University of Texas at Austin. Yan is currently visiting the U.S. Securities and Exchange Commission. The U.S. Securities and Exchange Commission disclaims responsibility for any private publication or statement of any SEC employee. This study expresses the authors' views and does not necessarily reflect those of the Commission, the Commissioners, or other members of the staff.

## Appendix

### Proof of Lemma 1

From (2) we have

$$\mu(s) = \overline{\mu} + (\mu(t) - \overline{\mu})e^{-\kappa(s-t)} + e^{-\kappa s}\sigma_\mu \int_t^s e^{\kappa\tau} dZ_D(\tau) \tag{A1}$$

and

$$\int_t^s \mu(u)du = \overline{\mu}(s-t) + (\mu(t) - \overline{\mu})B_\kappa(t,s) + \sigma_\mu \int_t^s B_\kappa(\tau,s)dZ_D(\tau) \tag{A2}$$

For notational convenience, define

$$B_\kappa(t,s) = \frac{1 - e^{-\kappa(s-t)}}{\kappa}, \tag{A3}$$

so we have

$$\int_t^s B_\kappa(t,u)du = \frac{1}{\kappa}[(s-t) - B_\kappa(t,s)]$$
$$\int_t^s B_\kappa(t,u)^2 du = \frac{1}{\kappa^2}[(s-t) - 2B_\kappa(t,s) + B_{2\kappa}(t,s)]$$

The time-$t$ price of the risk-free discount bond maturing at time $T$ is given by the discounted payoff under risk-neutral measure $\mathbb{Q}$.

$$\begin{aligned} P(t,T,r(t)) &= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\right] = \mathrm{E}_t\left[e^{-\int_t^T \frac{1}{2}\theta^2 du - \int_t^T \theta dZ_D(u) - \int_t^T r(u)du}\right] \\ &= \mathrm{E}_t\left[\frac{\pi(T)}{\pi(t)}\right] = \mathrm{E}_t\left[e^{-\delta(T-t)}\left(\frac{D_T}{D_t}\right)^{-\gamma}\right] \\ &= D_t^\gamma e^{-\delta(T-t)}\mathrm{E}_t\left[D_T^{-\gamma}\right] \end{aligned} \tag{A4}$$

Using Itô's Lemma, from (1) we have

$$d\ln D_s^{-\gamma} = -\gamma\left[\mu(s) - \frac{1}{2}\sigma_D^2\right]ds - \gamma\sigma_D dZ_D(s) \tag{A5}$$

and

$$\mathrm{E}_t\left[\ln D_T^{-\gamma}\right] = \ln D_t^{-\gamma} - \left(\overline{r} - \delta + \frac{1}{2}\gamma^2\sigma_D^2\right)(T-t) - (r(t) - \overline{r})B_\kappa(t,T) \tag{A6}$$

$$\begin{aligned}\mathbf{Var}_t\left[\ln D_T^{-\gamma}\right] &= \gamma^2\int_t^T (\sigma_D + \sigma_\mu B_\kappa(\tau,T))^2 d\tau \\ &= \gamma^2\left[\left(\sigma_D^2 + \frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)(T-t)\right. \\ &\quad \left. -\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{2\sigma_\mu^2}{\kappa^2}\right)B_\kappa(t,T) + \frac{\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t,T)\right].\end{aligned} \tag{A7}$$

Because $\ln D_s^{-\gamma}$ is normally distributed, we have

$$\begin{aligned}P(t,T,r(t)) &= D_t^\gamma e^{-\delta(T-t)}\exp\left\{\mathrm{E}_t\left[\ln D_T^{-\gamma}\right] + \frac{1}{2}\mathbf{Var}_t\left[\ln D_T^{-\gamma}\right]\right\} \\ &= e^{A(t,T) - B_\kappa(t,T)r(t)},\end{aligned} \tag{A8}$$

where

$$\begin{aligned}A(t,T) &= -\left[\overline{r} - \frac{1}{2}\gamma^2\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)\right](T-t) + \overline{r}B_\kappa(t,T) \\ &\quad -\frac{1}{2}\gamma^2\left[\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{2\sigma_\mu^2}{\kappa^2}\right)B_\kappa(t,T) - \frac{\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t,T)\right].\end{aligned}$$

The first order derivative is given by

$$\frac{\partial P(t,T,r(t))}{\partial T} = \left(\frac{\partial A(t,T)}{\partial T} - \frac{\partial B_\kappa(t,T)}{\partial T}r(t)\right)P(t,T,r(t)), \tag{A9}$$

where

$$\begin{aligned}\frac{\partial A(t,T)}{\partial T} &= \left(\gamma^2\sigma_D\sigma_\mu - \kappa\overline{r}\right)B_\kappa(t,T) + \frac{1}{2}\gamma^2\sigma_\mu^2 B_\kappa^2(t,T), \\ \frac{\partial B_\kappa(t,T)}{\partial T} &= e^{-\kappa(T-t)}.\end{aligned}$$

■

**Proof of Lemma 2**

The price of this dividend stream is given by

$$S(t) = \mathrm{E}_t\left[\int_t^\infty \frac{\pi(s)}{\pi(t)}D_s ds\right] \quad \left(= \mathrm{E}_t^{\mathbb{Q}}\left[\int_t^\infty e^{-\int_t^s r(u)du}D(s)ds\right]\right) \tag{B10}$$

$$= \mathrm{E}_t\left[\int_t^\infty e^{-\delta(s-t)}\left(\frac{D_s}{D_t}\right)^{-\gamma}D_s ds\right] \tag{B11}$$

$$= D_t^{\gamma} \int_t^{\infty} e^{-\delta(s-t)} \mathrm{E}_t\left[D_s^{1-\gamma}\right] ds \tag{B12}$$

From Ito's lemma, we have

$$dD_s^{1-\gamma} = (1-\gamma) D_s^{1-\gamma}\left[\mu(s) - \frac{1}{2}\gamma\sigma_D^2\right] ds + (1-\gamma) D_s^{1-\gamma}\sigma_D dZ_D \tag{B13}$$

Therefore

$$d\ln D_s^{1-\gamma} = (1-\gamma)\left[\mu(s) - \frac{1}{2}\sigma_D^2\right] ds + (1-\gamma)\sigma_D dZ_D(s) \tag{B14}$$

Therefore

$$\mathrm{E}_t\left[\ln D_s^{1-\gamma}\right] = \ln D_t^{1-\gamma} + (1-\gamma)\left[\left(\bar{\mu} - \frac{1}{2}\sigma_D^2\right)(s-t) + (\mu(t) - \bar{\mu}) B_\kappa(t,s)\right] \tag{B15}$$

$$\begin{aligned}
\mathbf{Var}_t\left[\ln D_s^{1-\gamma}\right] &= (1-\gamma)^2 \int_t^s (\sigma_D + \sigma_\mu B_\kappa(\tau,s))^2 d\tau \\
&= (1-\gamma)^2\left[\left(\sigma_D^2 + \frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)(s-t)\right. \\
&\quad \left. - \left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{2\sigma_\mu^2}{\kappa^2}\right) B_\kappa(t,s) + \frac{\sigma_\mu^2}{\kappa^2} B_{2\kappa}(t,T)\right]
\end{aligned} \tag{B16}$$

Since $\ln D_s^{1-\gamma}$ is conditionally normal, we have

$$S(t) = D_t^{\gamma} \int_t^{\infty} e^{-\delta(s-t)} \exp\left\{\mathrm{E}_t\left[\ln D_s^{1-\gamma}\right] + \frac{1}{2}\mathbf{Var}_t\left[\ln D_s^{1-\gamma}\right]\right\} ds \tag{B17}$$

$$= D_t \int_t^{\infty} \exp(\psi(t,s;r(t))) ds \tag{B18}$$

where

$$\begin{aligned}
\psi(t,s;r(t)) = &\left[-\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}\bar{r} + \frac{\gamma(1-\gamma)}{2}\sigma_D^2 + \frac{1}{2}(1-\gamma)^2\left(\sigma_D^2 + \frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)\right](s-t) \\
&+ \left[\frac{1-\gamma}{\gamma}(r(t) - \bar{r}) - \frac{1}{2}(1-\gamma)^2\left(\frac{2\sigma_D\sigma_\mu}{\kappa} + \frac{2\sigma_\mu^2}{\kappa^2}\right)\right] B_\kappa(t,s) \\
&+ \frac{1}{2}(1-\gamma)^2 \frac{\sigma_\mu^2}{\kappa^2} B_{2\kappa}(t,T])
\end{aligned}$$

■

**Proof of Lemma 3**

Easy calculation shows

$$\xi(s) = \overline{\xi} + (\xi(t) - \overline{\xi})e^{-\lambda(s-t)} + e^{-\lambda s}\sigma_\xi \int_t^s e^{\lambda\tau} dZ_K(\tau)$$

and

$$\int_t^s \xi(u)du = \overline{\xi}(s-t) + (\xi(t) - \overline{\xi})B_\lambda(t,s) + \sigma_\xi \int_t^s B_\lambda(\tau,s)dZ_K(\tau)$$

where

$$B_\lambda(t,s) = \frac{1 - e^{-\lambda(s-t)}}{\lambda}. \tag{C19}$$

The present value (or unlevered equity price) is given by

$$S_K(t) = \mathrm{E}_t^{\mathbb{Q}}\left[\int_t^\infty e^{-\int_t^s r(u)du} K(s)ds\right] \tag{C20}$$

$$= \mathrm{E}_t\left[\int_t^\infty \frac{\pi(s)}{\pi(t)} K(s)ds\right] \tag{C21}$$

$$= \mathrm{E}_t\left[\int_t^\infty e^{-\delta(s-t)}\left(\frac{D_s}{D_t}\right)^{-\gamma} K_s ds\right] \tag{C22}$$

$$= D_t^\gamma \int_t^\infty e^{-\delta(s-t)} \mathrm{E}_t\left[D_s^{-\gamma} K_s\right] ds \tag{C23}$$

Using Ito's lemma, we have

$$\begin{aligned} d\ln\left(D_s^{-\gamma}K_s\right) = &\left[(\beta-\gamma)\mu(t) + \xi(s) + \frac{1}{2}\gamma\sigma_D^2 - \frac{1}{2}\sigma_K^2\right]ds \\ &+ (\rho\sigma_K - \gamma\sigma_D)dZ_D(s) + \sigma_K\sqrt{1-\rho^2}dZ_K(s) \end{aligned} \tag{C24}$$

Therefore

$$\begin{aligned} \mathrm{E}_t\left[\ln\left(D_s^{-\gamma}K_s\right)\right] = &\ln(D_t^{-\gamma}K_t) + \left[(\beta-\gamma)\overline{\mu} + \overline{\xi} + \frac{1}{2}\gamma\sigma_D^2 - \frac{1}{2}\sigma_K^2\right](s-t) \\ &+ (\beta-\gamma)(\mu(t) - \overline{\mu})B_\kappa(t,s + (\xi(t) - \overline{\xi})B_\lambda(t,s) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}_t\left[\ln\left(D_s^{-\gamma}K_s\right)\right] = &\int_t^s \left(\left[(\beta-\gamma)\sigma_\mu B_\kappa(\tau,s) + \rho\sigma_K - \gamma\sigma_D\right]^2 + \left[\sigma_K\sqrt{1-\rho^2} + \sigma_\xi B_\lambda(\tau,s)\right]^2\right)d\tau \\ = &\left[(\rho\sigma_K - \gamma\sigma_D)^2 + \sigma_K^2(1-\rho^2) + 2(\beta-\gamma)(\rho\sigma_K - \gamma\sigma_D)\frac{\sigma_\mu}{\kappa}\right. \\ &\left. + 2\sqrt{1-\rho^2}\frac{\sigma_K\sigma_\xi}{\lambda} + \frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2} + \frac{\sigma_\xi^2}{\lambda^2}\right](s-t) \\ &- 2\left[(\beta-\gamma)(\rho\sigma_K - \gamma\sigma_D)\frac{\sigma_\mu}{\kappa} + \frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2}\right]B_\kappa(t,s) \\ &- 2\left[\sqrt{1-\rho^2}\frac{\sigma_K\sigma_\xi}{\lambda} + \frac{\sigma_\xi^2}{\lambda^2}\right]B_\lambda(t,s) + \frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t,s) + \frac{\sigma_\xi^2}{\lambda^2}B_{2\lambda}(t,s) \end{aligned}$$

Because $\ln\left(D_s^{-\gamma}K_s\right)$ is conditionally normal, we have

$$S_K(t) = D_t^{\gamma}\int_t^{\infty} e^{-\delta(s-t)}\exp\left\{\mathbf{E}_t\left[\ln\left(D_s^{-\gamma}K_s\right)\right] + \frac{1}{2}\mathrm{Var}_t\left[\ln\left(D_s^{-\gamma}K_s\right)\right]\right\}ds \tag{C25}$$

$$= K_t\int_t^{\infty}\exp(\psi_K(t,s;\mu(t),\xi(t)))ds \tag{C26}$$

where

$$\begin{aligned}\psi_K(t,s;\mu(t),\xi(t)) = \Bigg[ & -\delta + (\beta-\gamma)\overline{\mu} + \overline{\xi} + \frac{1}{2}\gamma\sigma_D^2 - \frac{1}{2}\sigma_K^2 + \frac{1}{2}(\rho\sigma_K - \gamma\sigma_D)^2 \\ & + \frac{1}{2}\sigma_K^2(1-\rho^2) + (\beta-\gamma)(\rho\sigma_K - \gamma\sigma_D)\frac{\sigma_\mu}{\kappa} + \sqrt{1-\rho^2}\,\frac{\sigma_K\sigma_\xi}{\lambda} \\ & + \frac{1}{2}\frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2} + \frac{1}{2}\frac{\sigma_\xi^2}{\lambda^2}\Bigg](s-t) + (\beta-\gamma)(\mu(t)-\overline{\mu})B_\kappa(t,s) \\ & + (\xi(t)-\overline{\xi})B_\lambda(t,s) - \left[(\beta-\gamma)(\rho\sigma_K - \gamma\sigma_D)\frac{\sigma_\mu}{\kappa} + \frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2}\right]B_\kappa(t,s) \\ & - \left[\sqrt{1-\rho^2}\,\frac{\sigma_K\sigma_\xi}{\lambda} + \frac{\sigma_\xi^2}{\lambda^2}\right]B_\lambda(t,s) \\ & + \frac{1}{2}\frac{(\beta-\gamma)^2\sigma_\mu^2}{\kappa^2}B_{2\kappa}(t,s) + \frac{1}{2}\frac{\sigma_\xi^2}{\lambda^2}B_{2\lambda}(t,s)\end{aligned}$$

■

**Proof of Lemma 4**

Under the risk-neutral measure $\mathbb{Q}$ the cash flow dynamics is

$$\frac{dK(t)}{K(t)} = (\beta\mu(t) + \xi(t) - \gamma\rho\sigma_D\sigma_K)dt + \sigma_K\rho dZ_D^{\mathbb{Q}}(t) + \sigma_K\sqrt{1-\rho^2}dZ_K^{\mathbb{Q}}(t), \tag{D27}$$

$$d\xi(t) = \lambda(\overline{\xi} - \xi(t))dt + \sigma_\xi dZ_K^{\mathbb{Q}}(t), \tag{D28}$$

where $dZ_K^{\mathbb{Q}}(t) = dZ_K(t)$ because $dZ_K(t)$ is independent of $dZ_D(t)$. Using change of numerair:

$$E^{\mathbb{F}_\mathbf{T}}\left[1_{\{A\}}\right] = \mathrm{E}^{\mathbb{Q}}\left[\frac{e^{-\int_0^T r(t)dt}}{P(T,r)}1_{\{A\}}\right], \tag{D29}$$

we have

$$dZ_D^{\mathbb{Q}}(t) = dZ_D^{\mathbb{F}_\mathbf{T}}(t) + \beta(t)dt \tag{D30}$$

where

$$\beta(t) = \frac{\gamma\sigma_\mu P_r(t,T,r(t))}{P(t,T,r(t))} = -\gamma\sigma_\mu B_\kappa(t,T). \tag{D31}$$

Therefore under the forward risk-neutral measure $\mathbb{F}_{\mathbf{T}}$, the cash flow dynamics is

$$\frac{dK(t)}{K(t)} = (\beta\mu(t) + \xi(t) - \gamma\rho\sigma_D\sigma_K - \gamma\rho\sigma_\mu\sigma_K B_\kappa(t,T))dt \tag{D32}$$

$$+\sigma_K\rho dZ_D^{\mathbb{F}_{\mathbf{T}}}(t) + \sigma_K\sqrt{1-\rho^2}dZ_K^{\mathbb{F}_{\mathbf{T}}}(t), \tag{D33}$$

$$d\xi(t) = \lambda(\overline{\xi} - \xi(t))dt + \sigma_\xi dZ_K^{\mathbb{F}_{\mathbf{T}}}(t). \tag{D34}$$

Let $X(t) = \ln(K(t)/c)$, we have

$$dX(t) = \Big(\beta\mu(t) + \xi(t) - \frac{1}{2}\sigma_K^2 - \gamma\rho\sigma_D\sigma_K - \gamma\rho\sigma_\mu\sigma_K B_\kappa(t,T)\Big)dt + \sigma_K\rho dZ_D^{\mathbb{F}_{\mathbf{T}}}(t)$$
$$+ \sigma_K\sqrt{1-\rho^2}dZ_K^{\mathbb{F}_{\mathbf{T}}}(t).$$

Therefore

$$X(s) = X(t) + \beta\int_{\mathrm{t}}^{\mathrm{s}}\mu(u)du + \int_t^s\xi(u)du - \Big(\frac{1}{2}\sigma_K^2 + \gamma\rho\sigma_D\sigma_K t)(s-t) - \frac{\gamma\rho\sigma_\mu\sigma_K}{\kappa}$$
$$\times\Big[(s-t) - e^{-\kappa(T-s)}B_\kappa(t,s)\Big] + \sigma_K\Big[\int_t^s\rho dZ_D^{\mathbb{F}_{\mathbf{T}}}(u) + \sqrt{1-\rho^2}\int_t^s dZ_K^{\mathbb{F}_{\mathbf{T}}}(u)\Big].$$

Recall

$$\mu(s) = \overline{\mu} + (\mu(t) - \overline{\mu})e^{-\kappa(s-t)} - \gamma\sigma_D\sigma_\mu B_\kappa(t,s)$$
$$-\frac{\gamma\sigma_\mu^2}{\kappa}\Big[B_\kappa(t,s) - e^{-\kappa(T-s)}B_{2\kappa}(t,s)\Big] + e^{-\kappa s}\sigma_\mu\int_t^s e^{\kappa\tau}dZ_D^{\mathbb{F}_{\mathbf{T}}}(\tau),$$

therefore

$$\int_t^s\mu(u)du = \overline{\mu}(s-t) + (\mu(t) - \overline{\mu})B_\kappa(t,s) - \frac{\gamma\sigma_D\sigma_\mu}{\kappa}\Big[(s-t) - B_\kappa(t,s)\Big] - \frac{\gamma\sigma_\mu^2}{\kappa}$$
$$\Big[\frac{1}{\kappa}\Big((s-t) - B_\kappa(t,s)\Big) - \frac{1}{2}e^{-\kappa(T-s)}B_\kappa^2(t,s)\Big] + \sigma_\mu\int_t^s B_\kappa(u,s)dZ_D^{\mathbb{F}_{\mathbf{T}}}(u)$$

We also have

$$\int_t^s\xi(u)du = \overline{\xi}(s-t) + (\xi(t) - \overline{\xi})B_\lambda(t,s) + \sigma_\xi\int_t^s B_\lambda(\tau,s)dZ_K(\tau).$$

Therefore we have the conditional mean

$$E_t^{\mathbb{F}_{\mathbf{T}}}[X_s] = X(t) + \Big[\beta\overline{\mu} + \overline{\xi} - \beta\frac{\gamma\sigma_D\sigma_\mu}{\kappa} - \beta\frac{\gamma\sigma_\mu^2}{\kappa^2} - \frac{\gamma\rho\sigma_\mu\sigma_K}{\kappa} - \frac{1}{2}\sigma_K^2 - \gamma\rho\sigma_D\sigma_K\Big](s-t)$$
$$+\beta(\mu(t)) - \overline{\mu})B_\kappa(t,s) + (\xi(t) - \overline{\xi})B_\lambda(t,s) + \beta\frac{\gamma\sigma_D\sigma_\mu}{\kappa}B_\kappa(t,s)$$
$$+\beta\frac{\gamma\sigma_\mu^2}{\kappa^2}B_\kappa(t,s) + \frac{\gamma\rho\sigma_\mu\sigma_K}{\kappa}e^{-\kappa(T-s)}B_\kappa(t,s) + \beta\frac{\gamma\sigma_\mu^2}{2\kappa}e^{-\kappa(T-s)}B_\kappa^2(t,s)$$

and conditional covariance, where $s_1 < s_2$,

$$\begin{aligned}\mathbf{Cov}_t^{\mathbb{F}_\mathbf{T}}[X_{s_1}, X_{s_2}] &= \int_t^{s_1} \Big[\big(\rho\sigma_K + \beta\sigma_\mu B_\kappa(\tau, s_1)\big)\big(\rho\sigma_K + \beta\sigma_\mu B_\kappa(\tau, s_2)\big) \\ &\quad + \big(\sigma_K\sqrt{1-\rho^2} + \sigma_\xi B_\lambda(\tau, s_1)\big)\big(\sigma_K\sqrt{1-\rho^2} + \sigma_\xi B_\lambda(\tau, s_2)\big)\Big] d\tau \\ &= \int_t^{s_1} \Big[\sigma_K^2 + \beta\rho\sigma_K\sigma_\mu\big(B_\kappa(\tau, s_1) + B_\kappa(\tau, s_2)\big) + \beta^2\sigma_\mu^2 B_\kappa(\tau, s_1)B_\kappa(\tau, s_2) \\ &\quad + \sigma_K\sigma_\xi\sqrt{1-\rho^2}\big(B_\lambda(\tau, s_1) + \mathrm{B}_\lambda(\tau, s_2)\big) + \sigma_\xi^2 B_\lambda(\tau, s_1)B_\lambda(\tau, s_2)\Big] d\tau \\ &= \left[\sigma_K^2 + \frac{2\beta\rho\sigma_K\sigma_\mu}{\kappa} + \frac{\beta^2\sigma_\mu^2}{\kappa^2} + \frac{2\sigma_K\sigma_\xi\sqrt{1-\rho^2}}{\lambda} + \frac{\sigma_\xi^2}{\lambda^2}\right](s_1 - t) \\ &\quad - \left(1 + e^{-\kappa(s_2 - s_1)}\right)\left[\frac{\beta\rho\sigma_K\sigma_\mu}{\kappa} + \frac{\beta^2\sigma_\mu^2}{\kappa^2}\right] B_\kappa(t, s_1) \\ &\quad - \left(1 + e^{-\lambda(s_2 - s_1)}\right)\left[\frac{\sigma_K\sigma_\xi\sqrt{1-\rho^2}}{\lambda} + \frac{\sigma_\xi^2}{\lambda^2}\right] B_\lambda(t, s_1) \\ &\quad + e^{-\kappa(s_2 - s_1)}\frac{\beta^2\sigma_\mu^2}{\kappa^2} B_{2\kappa}(t, s_1) + e^{-\lambda(s_2 - s_1)}\frac{\sigma_\xi^2}{\lambda^2} B_{2\lambda}(t, s_1)\end{aligned}$$

Under the physical measure $\mathbb{P}$, the expected mean is

$$E_t^{\mathbb{P}}[X_s] = X(t) + \left[\beta\overline{\mu} + \overline{\xi} - \frac{1}{2}\sigma_K^2 - \gamma\rho\sigma_D\sigma_K\right](s - t) + \beta(\mu(t) - \overline{\mu})B_\kappa(t, s) + (\xi(t) - \overline{\xi})B_\lambda(t, s)$$

Expected variances and covariances are the same as in the risk-neutral $T$-forward measure. ■

**Proof of Proposition 1**

The expected payoff stream of this coupon bond is

$$\begin{aligned} g(t) &= c \cdot \mathbf{1}(t \le T) \cdot \mathbf{1}(t < \tau) + F \cdot \delta(t - T) \cdot \mathbf{1}(t < \tau) \\ &\quad + w(\mu_t)F \cdot \delta(t - \tau) \cdot \mathbf{1}(t \le T). \end{aligned} \tag{E35}$$

Assume the recovery rate is proportional to economy growth rate,

$$w(\mu_t) = a + b\mu_t, \tag{E36}$$

where $b \ge 0$.

$$\begin{aligned} DV &= \mathrm{E}^{\mathbb{Q}}\left[\int_0^T e^{-\int_0^t r(s)ds} g(t)dt\right] = c\int_0^T \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^t r(s)ds}\mathbf{1}(t < \tau)\right]dt \\ &\quad + F \cdot \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\mathbf{1}(T < \tau)\right] + F\int_0^T \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^t r(s)ds} w(\mu_t)\delta(t - \tau)\right]dt \\ &= c\int_0^T P(0, t, r(0))\mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\mathbf{1}(t < \tau)]dt + F \cdot P(0, T, r(0)) \cdot \mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\mathbf{1}(T < \tau)] \\ &\quad + F\int_0^T P(0, t, r(0))\mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[w(\mu_t)]\mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\delta(t - \tau)]dt = FV - EL \end{aligned}$$

where

$$FV = c\int_0^T P(0,t,r(0))dt + F\cdot P(0,T,r(0)),$$

is the value of a default risk-free bond with identical payment structure, and

$$\begin{aligned} EL = c\int_0^T P(0,t,r(0))(t)dt + (1-a-b\mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)])F\cdot P(0,T,r(0))\cdot\Gamma(T) \\ + F\int_0^T\left(P_t(0,t,r(0))(a+b\mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)]) + P(0,t,r(0))b\frac{\partial\mathrm{E}_0^{\mathbb{F}_\mathbf{T}[\mu(t)]}}{\partial t}\right)\Gamma(t)dt \end{aligned}$$

is the expected loss of the risky bond, where $\Gamma(t)\equiv\mathrm{E}^{\mathbb{F}_\mathbf{T}}[\mathbf{1}(\tau\le t)]$ is the cumulative distribution function of $\tau$ in the risk neutral $\boldsymbol{T}$-forward measure (which represetns the probability that the firm will default before time $t$ ), its density function is $f(t)\equiv E^{\mathbb{F}_\mathbf{T}}[\delta(t-\tau)]$, and $P_t(0,t,r(0)) = \partial P_t(0,t,r(0))/\partial t$.

From (2) we have

$$\begin{aligned} \mu(s) = \overline{\mu} + (\mu(t)-\overline{\mu})e^{-\kappa(s-t)} - \gamma\sigma_D\sigma_\mu B_\kappa(t,s) \\ -\frac{\gamma\sigma_\mu^2}{\kappa}\left[B_\kappa(t,s) - e^{-\kappa(T-s)}B_{2\kappa}(t,s)\right] + e^{-\kappa s}\sigma_\mu\int_t^s e^{\kappa\tau}dZ_D^{\mathbb{F}_\mathbf{T}(\tau)} \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)] &= \overline{\mu} + (\mu(0)-\overline{\mu})e^{-\kappa t} - \gamma\sigma_D\sigma_\mu B_\kappa(0,t) \\ &\quad -\frac{\gamma\sigma_\mu^2}{\kappa}\left[B_\kappa(0,t) - e^{-\kappa(T-t)}B_{2\kappa}(0,t)\right] \\ \frac{\partial\mathrm{E}_0^{\mathbb{F}_\mathbf{T}}[\mu(t)]}{\partial t} &= -\left[\kappa(\mu(0)-\overline{\mu}) + \gamma\sigma_D\sigma_\mu + \frac{\gamma\sigma_\mu^2}{\kappa}\right]e^{-\kappa t} \\ &\quad + \gamma\sigma_\mu^2 e^{-\kappa(T-t)}\frac{1+e^{-2\kappa t}}{2\kappa} \end{aligned}$$

■

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