Online Appendix for "Slow Information Diffusion and the Inertial Behavior of Durable Consumption"

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1. Solving Bernanke's Adjustment Costs Model

Given the utility function

$$u\left(c_t, k_t, k_{t-1}\right) = -\frac{1}{2}\left(\bar{c} - c_t\right)^2 - \frac{\varrho}{2}\left(\bar{k} - k_t\right)^2 - \frac{\vartheta}{2}\left(k_t - k_{t-1}\right)^2 \tag{1}$$

and the budget constraints

$$a_{t+1} = Ra_t + (1-\delta)\,k_{t-1} - k_t + y_t - c_t, \tag{2}$$

we can solve for the following decision rules of nondurables and durables following the same procedure used in Bernanke (1985):

$$c_t = G_c\left(a_t + G_k k_{t-1} + G_y y_t\right) + g_0, \tag{3}$$

$$k_t = x_1 k_{t-1} + \frac{x_1\left(1-\beta\left(1-\delta\right)\right)}{d\left(1-x_2^{-1}\right)} c_t + h_0, \tag{4}$$

where g_0 and h_0 are irrelevant constant terms, x_1 and x_2 satisfying $x_1 + x_2 = \frac{\beta\varrho + (1+\beta)\vartheta}{\beta\vartheta}$ and $x_1 x_2 = \frac{1}{\beta}$ are two real eigenvalues (suppose $x_1 < x_2$ without loss of generality) for the second-order stochastic difference equation:

$$k_{t-1} + hk_t + \beta E_t\left[k_{t+1}\right] = 0,$$

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and

$$G_y = \frac{1}{R-1}, \tag{5}$$

$$G_k = \beta(1-\delta) + (\beta(1-\delta) - 1)\frac{x_1}{R-x_1}, \tag{6}$$

$$G_c = (R-1)\left(1 + (1-\beta(1-\delta))^2 \frac{R}{R-x_1}\frac{x_1}{\vartheta\left(1-x_2^{-1}\right)}\right)^{-1}. \tag{7}$$

Note that as ϑ goes to 0, (3) and (4) reduces to (7) and (8) in the main text because $\lim_{\vartheta\to 0} x_1 = 0$ and $\lim_{\vartheta\to 0} \frac{x_1}{\vartheta\left(1-x_2^{-1}\right)} = \frac{1}{\beta\varrho}$.

To obtain the explicit dynamics of nondurables and durables consumption in this model, we define a new state variable, permanent income s_t, as $s_t = a_t + G_k k_{t-1} + G_y y_t$, and reformulate the original budget constraint (2) as

$$s_{t+1} = s_t + G_y \varepsilon_{t+1}. \tag{8}$$

after using (3) and (4). We can then obtain the dynamics of (c_t, k_t), (53) and (54) of the text, by combining (3), (4), with (8).

2. Solving the CARA Model under SID

In the CARA model with the utility function: $u(c_t, k_t) = -\frac{1}{\alpha_c}\exp(-\alpha_c c_t) - \frac{\varrho}{\alpha_k}\exp(-\alpha_k k_t)$, where $\alpha_c > 0$ and $\alpha_k > 0$, following the same procedure adopted in Caballero (1990) and using the same budget constraint specified in Section 6.3, we can readily solve for the decision rules for both nondurable and durable consumption under full information as follows:[1]

$$c_t = H_c s_t + \Omega_c + \Pi_c, \tag{9}$$

$$k_t = \Omega_k + \frac{\alpha_c}{\alpha_k} c_t, \tag{10}$$

where $s_t = a_t + \frac{1-\delta}{R}k_{t-1} + \frac{1}{R-1}y_t$, $H_c = (R-1)\left[1 + \frac{(R+\delta-1)^2}{R\varrho}\right]^{-1}$, $\Omega_c = -\frac{R+\delta-1}{R}\left[1 + \frac{(R+\delta-1)^2}{R\varrho}\right]^{-1}\Omega_k$, $\Omega_k = -\frac{1}{\alpha_k}\ln\left(\frac{R+\delta-1}{\varrho}\right)$, $\Pi_c = -\frac{1}{R-1}\Phi$, and

$$\Phi = \frac{1}{\alpha_c}\ln(\beta R) + \frac{1}{2}\alpha_c\left[1 + \frac{(R+\delta-1)^2}{\varrho R}\right]^{-2}\omega^2. \tag{11}$$

Comparing (7) (and (8)) with (9) (and (10)), it is clear that the MPC out of permanent income in the model with a quadratic utility function and the model with a CARA utility function are the same. Consequently, the two models lead to the same stochastic properties of the joint dynamics of nondurable and durable consumption.

[1]Note that here we set $\frac{\alpha_c}{\alpha_k} = \frac{R+\delta-1}{\varrho}$ such that the ratios of the marginal propensity to consume (MPC) in the durable and nondurable consumption functions are the same as in both the quadratic and CARA-PIH models.

Incorporating the SID assumption into the CARA model, we formulate the optimization problem for the typical household facing state uncertainty:

$$v\left(\widehat{s}_t\right)=\max_{\left\{c_t,k_{t+1}\right\}}E_t\left[u\left(c_t,k_t\right)+\beta v\left(\widehat{s}_{t+1}\right)\right]$$

subject to

$$\widehat{s}_{t+1}=R\widehat{s}_t-c_t-\left(1-\beta\left(1-\delta\right)\right)k_t+\eta_{t+1}, \tag{12}$$

where

$$\eta_{t+1}=\theta\left[\left(\frac{\zeta_{t+1}}{1-(1-\theta)R\cdot L}\right)+\left(\xi_{t+1}-\frac{\theta R\xi_t}{1-(1-\theta)R\cdot L}\right)\right], \tag{13}$$

$$\omega_\eta^2=\mathrm{var}\left(\eta_{t+1}\right)=\frac{\theta}{1-\left(1-\theta\right)R^2}\omega_\zeta^2>\omega_\zeta^2,\text{ for }\theta<1, \tag{14}$$

and given $\widehat{a}_0$.[2] Solving this Bellman equation yields the following consumption and durable accumulation functions:

$$c_t=H_c\widehat{s}_t+\Omega_c+\Pi_c, \tag{15}$$

$$k_t=\Omega_k+\frac{\alpha_c}{\alpha_k}c_t. \tag{16}$$

Given the original budget constraint and the two decision rules, the expression for individual saving, $d_t\left(\equiv\left(R-1\right)a_t+y_t-c_t-\left(k_t-\left(1-\delta\right)k_{t-1}\right)\right)$, can be written as:

$$d_t=\left[1-\frac{1+\left(R+\delta-1\right)/\varrho}{1+\left(R+\delta-1\right)^2/\left(\varrho R\right)}\right]\eta_t+\left(R-1\right)\left(s_t-\widehat{s}_t\right)+\left[\frac{R}{H_c}-\left(1+\frac{R+\delta-1}{\varrho}\right)\right]\Phi,$$

where $\eta_t=\theta\left[\left(\frac{\zeta_t}{1-(1-\theta)R\cdot L}\right)+\left(\xi_t-\frac{\theta R\xi_{t-1}}{1-(1-\theta)R\cdot L}\right)\right]$ and $s_t-\widehat{s}_t=\frac{(1-\theta)\zeta_t}{1-(1-\theta)R\cdot L}-\frac{\theta\xi_t}{1-(1-\theta)R\cdot L}$. Following the same aggregation procedure presented in the last section, aggregating across all consumers yields the expression for aggregate saving, Expression (59) in the text. Using (59), the expression for Φ:

$$\Phi=\frac{1}{\alpha_c}\ln\left(\beta R\right)+\frac{1}{2}\alpha_c\left[1+\frac{\left(R+\delta-1\right)^2}{\varrho R}\right]^{-2}\omega^2,$$

and the corresponding expression for aggregate saving in the benchmark model in which

$$\overline{d}_t=\left[1-\frac{1+\left(R+\delta-1\right)/\varrho}{1+\left(R+\delta-1\right)^2/\left(\varrho R\right)}\right]\frac{\theta\zeta_t}{1-(1-\theta)R\cdot L}+\frac{\left(R-1\right)\left(1-\theta\right)\zeta_t}{1-(1-\theta)R\cdot L},$$

we can obtain (60) in the text.

[2] If RI is considered as the microfoundation for the slow information diffusion, the results from the CARA case are valid only approximately when the capacity is not too low.

3. Fixed Cost and Infrequent Adjustment

Consider the SID model proposed in Section 6.4. Here for simplicity we only consider the original Mankiw model (no nondurable goods) in which the utility function $u(k_t) = -\frac{1}{2}\left(\bar{k} - k_t\right)^2$. Under SID, the agent adjusts optimal consumption plans in every period but the adjustments are incomplete. In this case, the welfare loss due to incomplete adjustment is

$$v^1 = \min E_t\left[\frac{1}{2}\sum_{j=t}^{\infty}\beta^{j-t}(k_j - k_j^*)^2\right],$$

where $E_t[\cdot]$ is formed using processed information and is subject to noisy observations described in Section 6.3, and $k_j^* = H_k s_j$ is the first-best FI-RE plan. As shown in Section 4.2, the optimal consumption plan under SID can be written as $k_t = H_k E_t[s_t]$, where $E_t[s_t]$ is the perceived state variable. Substituting the optimal rule under SID into the objective function, the welfare loss due to incomplete adjustments can be rewritten as

$$v^1 = \frac{1}{2}\left[\sum_{j=t}^{\infty}\beta^{s-t}E_t(k_j - H_k E_j[s_j])^2\right] = \frac{1}{2}\frac{H_k^2\sigma^2}{1-\beta},$$

where $j \geq t$, and $\sigma^2 = \operatorname{var}_j(s_j) = E_j(k_j - H_k E_j[s_j])^2$ is the steady state conditional variance from the Kalman filter. Here we assume that the typical consumer with imperfect state observations faces fixed costs ($F > 0$) in each period for adjusting optimal plans; the present value of fixed costs is then $\frac{\beta F}{1-\beta}$.

We now consider a model of infrequent adjustment in durable consumption. The key assumption of this model is that consumers are inattentive in the sense that every period they update the information about their permanent income with some probability and thus adjust optimal plans infrequently; in other words, only a fraction of them update their information and make optimal adjustments in any period. Following the literature, we assume that the exogenous probability at which the typical consumer updates his expectations and re-optimizes in any given period is π, independent of the length of time since the optimal plan was set. The consumer sets his optimal consumption plan at t to minimize a quadratic loss function that depends on the difference between the consumer's actual consumption plan at period t, k_t, and his first-best instantaneously-adjusted plan k_t^*. If the consumer chooses to adjust at period t, he sets optimal consumption to minimize

$$\frac{1}{2}E_t\left[\sum_{j=t}^{\infty}\beta^{j-t}(k_j - k_j^*)^2\right],$$

where $E_t[\cdot]$ is formed using all available information. The following diagram illustrates the evolution of infrequent adjustments over time.[3]

$$\text{If adjust at } t \begin{cases} \pi\text{: adjust at } t+1 \\ 1-\pi\text{: not adjust at } t+1 \begin{cases} \pi\text{: adjust at } t+2 \\ 1-\pi\text{: not adjust at } t+2 \begin{cases} \pi\text{: adjust at } t+3 \\ 1-\pi\text{: not adjust at } t+3 \end{cases} \cdots \end{cases} \end{cases}$$

Therefore, the present discounted welfare losses if the agent adjusts at time t (and re-optimizes with the same probability π at $t+1$), v^2, can be written as

$$v^2 = E_t\left[\frac{1}{2}\sum_{j=t}^{\infty}\left(\left(1-\pi\right)\beta\right)^{j-t}\left(k_t - k_j^*\right)^2\right] + \left[\sum_{j=t+1}^{\infty}\left(\left(1-\pi\right)^{j-t-1}\beta^{j-t}\right)\right]\pi\left(v^2+F\right). \tag{17}$$

The first term in (17) measures the welfare losses due to deviations of actual plans from desired (first-best) plans, and the losses are discounted by the discount factor $\left(\beta^{j-t}\right)$ and the probability that k_t will still be set in period j $\left(\left(1-\pi\right)^{j-t}\right)$; the second term represents the value of adjusting consumption plans at period j ($j > t$) and continuing the procedure. Solving (17) gives

$$\left(1-\beta\right)v^2 = \frac{1}{2}\left[1-\beta\left(1-\pi\right)\right]E_t\left[\sum_{j=t}^{\infty}\left(\left(1-\pi\right)\beta\right)^{j-t}\left(k_t-k_j^*\right)^2\right] + \pi\beta F;$$

the first order condition with respect to k_t means that

$$\begin{aligned} k_t &= \left[1-\beta\left(1-\pi\right)\right]\sum_{j=t}^{\infty}\left(\beta\left(1-\pi\right)\right)^{j-t}E_t\left[k_j^*\right], j \geq t \\ &= H_k\widehat{s}_t \equiv k_t^*, \text{ for any } t \geq 0, \end{aligned}$$

where we use the facts that under noisy signals and slow learning $k_j^* = H_k\widehat{s}_j$, $s_{j+1} = s_j + \zeta_{j+1}$, and $\widehat{s}_{j+1} = \widehat{s}_j + \eta_{j+1}$. We can therefore calculate that

$$\left(1-\beta\right)v^2 = \frac{1}{2}H_k^2\left[\frac{\beta\left(1-\pi\right)}{1-\beta\left(1-\pi\right)}\omega_\zeta^2 + \sigma^2\right] + \pi\beta F,$$

which implies that

$$v^2 = \frac{1}{2}\frac{H_k^2}{1-\beta}\left[\frac{\beta\left(1-\pi\right)}{1-\beta\left(1-\pi\right)}\omega_\zeta^2+\sigma^2\right] + \frac{\pi\beta}{1-\beta}F. \tag{18}$$

In this case, if the agent adjusts in every period v^2 reduces to $v^1 + \frac{\beta F}{1-\beta}$.[4]

Furthermore, if we allow for endogenous choice of the probability π, the first-order condition

[3] Kiley (2000) applied a similar idea to a firm optimization problem.
[4] Note that we have imposed the restriction that $\beta R = 1$.

for (18) implies that the optimal probability is

$$\pi^* = \frac{H_k \omega_\zeta}{\beta\sqrt{2F}} + \frac{\beta - 1}{\beta}, \tag{19}$$

which means that the optimal frequency of adjustment is increasing in the volatility of the innovation to permanent income (ω_ζ) and decreasing in the fixed cost F. Therefore, we have the following key result: If the fixed cost is small enough, i.e.,

$$F < F^* \equiv \frac{1}{2} H_k^2 \omega_\zeta^2, \tag{20}$$

it is optimal for the inattentive consumer to adjust in each period. The result can be obtained by substituting (18) and (19) into $v^2 < v^1 + \frac{\beta F}{1-\beta}$. Note that it is optimal for the consumer to adjust infrequently if and only if $v^2 < v^1 + \frac{\beta F}{1-\beta}$.

4. Aggregate Consumption under Infrequent Adjustment

Denote

$$\widetilde{k}_{i,t} = H_k \Delta \widehat{s}_t,$$

the optimal durables stock chosen by a household i who updated expectations about permanent income in (the current) period t.[5] Hence, this consumer's actual consumption equals the optimal levels of consumption chosen:

$$k_{i,t} = \widetilde{k}_{i,t}.$$

Households who do not update their expectations in period $t+1$ consume

$$k_{i,t+1} = k_{i,t} = \widetilde{k}_{i,t}$$

until they update their expectations; we assume updating happens with the probability π.[6] Nondurable and durable consumption per capita in period t that would prevail if all consumers updated their expectations are

$$\Delta \widetilde{K}_t = \int_0^1 \Delta \widetilde{k}_{i,t} di.$$

[5] As shown in Section 4.2,

$$\Delta \widehat{s}_t = \theta \left[\frac{\zeta_t}{1-(1-\theta)R \cdot L} + \left(\xi_t - \frac{\theta R \xi_{t-1}}{1-(1-\theta)R \cdot L} \right) \right].$$

[6] This specification differs from Reis (2006); in his model, consumption can evolve deterministically over the "planning period." Given any distaste for intertemporal variance of consumption combined with perfect capital markets, such deterministic variation would be suboptimal anyway.

Because the set of consumers who choose to update is randomly selected from the continuum of agents, the mean consumption of those consumers who choose to update can be written as

$$\Delta K_t^{\pi} = \int_0^1 \pi_{i,t} \Delta \widetilde{k}_{i,t} di = \pi \Delta \widetilde{K}_t.$$

If this result holds in every past period, it leads to the following expressions for per capita non-durable and durable consumption[7]

$$\Delta K_t^{ia} = \pi \sum_{j=0}^{\infty} (1-\pi)^j \, \Delta \widetilde{K}_{t-j}. \tag{21}$$

Aggregating the change in individual consumption across all consumers,

$$\Delta \widetilde{k}_{i,t} = \theta H_k \left[\frac{\zeta_t}{1-(1-\theta)R \cdot L} + \left(\xi_t - \frac{\theta R \xi_{t-1}}{1-(1-\theta)R \cdot L} \right) \right], \tag{22}$$

we obtain

$$\Delta \widetilde{K}_t = \theta H_k \frac{\zeta_t}{1-(1-\theta)R \cdot L}$$

which means that

$$\Delta K_t^{ia} = \frac{\pi \theta H_k \zeta_t}{(1-(1-\pi) \cdot L)\,(1-(1-\theta)R \cdot L)}. \tag{23}$$

References

[1] Bernanke, B. (1985), "Adjustment Costs, Durables, and Aggregate Consumption," *Journal of Monetary Economics* 15, 41-68.

[2] Caballero, R.J. (1990), "Consumption Puzzles and Precautionary Savings," *Journal of Monetary Economics* 25, 113-136.

[3] Carroll, C.D. and J. Slacalek (2006), "Sticky Expectations and Consumption Dynamics," mimeo, Johns Hopkins University.

[4] Kiley, M. (2000), "Endogenous Price Stickiness and Business Cycle Persistence," *Journal of Money, Credit and Banking* 32, 28-53.

[5] Reis, R. (2006), "Inattentive Consumers," *Journal of Monetary Economics* 53, 1761-1800.

[7] See Carroll and Slacalek (2006) for the derivation.