Topic 6: Stochastic Growth and Real Business Cycles II

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Output Persistence in the Standard RBC Model

- Given that $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, we have (see the last lecture note):

$$y_t = 0.04k_t + 1.924a_t. \tag{1}$$

where we use the fact that $\alpha = 0.36$ and $l_t = -0.5k_t + 1.44a_t$. Using $k_{t+1} = 0.94k_t + 0.158a_t$,i.e.,

$$k_t = \frac{0.158a_t}{1 - 0.94L},$$

the output process can be written as

$$y_t = \frac{0.006a_t}{1 - 0.94L} + 1.924a_t. \tag{2}$$

- (2) clearly shows that output dynamics are determined primarily by *impulse dynamics* (i.e., the dynamic impacts from a_t), with little contribution from *the endogenous propagation mechanism* (determined by capital accumulation and labor supply).

Model vs. Data

- Since propagation mechanisms are weak, the standard model must rely on external sources of dynamics in order to replicate the univariate dynamics of real GDP.
- In the above model, technology shocks evolve as a persistent AR(1) about trend, so output is approximately AR(1) with the same AR persistence coefficient as technology shocks. This model prediction contradicts the empirical evidence because the real GDP time series in the U.S. is well approximated by a second-order AR process, i.e., AR(2) process.

Hodrick-Prescott Filter (HP Filter)

- It is important to decompose *trend-cycle* in macroeconomics. The basic idea of the HP filter is to decompose the economic series of interest (e.g., the log of GDP) into the sum of a *trend* (growth component, g_t) and a *cycle* (transitory deviation from trend, c_t):

$$y_t = g_t + c_t. \tag{3}$$

- The HP filter extracts the trend g_t by solving the following penalty problem:

$$\min_{\{g_t\}} \underbrace{\sum_{t=1}^{T} (y_t - g_t)^2}_{\text{goodness of fit}} + \lambda \underbrace{\sum_{t=2}^{T-1} \left[(g_{t+1} - g_t) - (g_t - g_{t-1}) \right]^2}_{\text{penalty for roughness}}, \tag{4}$$

- In (4), the HP filter maximizes the fit of the trend to the series, i.e., minimize the cycle component in (3), while minimizing the changes in the trend's slope.
- λ controls the smoothess of the adjusted trend series. As $\lambda \to 0$, the trend approximates the actual series, when $\lambda \to \infty$, the trend becomes linear.
- For quarterly data, a value of $\lambda = 1600$ is reasonable. For annual data, $\lambda = 6$ or 7.

Blanchard-Kahn Method (The Eigensystem Method)

- Given the log-linearized system, we can use the Blanchard-Kahn method (1980) to solve for the optimal solutions numerically. For the generalized method to deal with more general RE linear models and the continuous-time case, see Sims (2000).
- Specifically, consider

$$0 = -k_{t+1} + \alpha_1 k_t + \alpha_2 \lambda_t + \alpha_3 a_t, \tag{5}$$

$$0 = E_t \left[-\lambda_t + \alpha_4 k_{+1} + \alpha_5 \lambda_{t+1} + \alpha_6 a_{t+1} \right], \tag{6}$$

where $\lambda_t = -c_t$. In the model, k_t is the endogenous state variable (it is also called "pre-determined" variable as its current value depends on its lagged value), a_t is the exogenous state variable, λ_t (i.e., c_t) is the control variable (it is also called "jump" variable). The system can be rewritten as:

$$\begin{bmatrix} 1 & 0 \\ \alpha_4 & \alpha_5 \end{bmatrix} E_t \begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 \\ -\alpha_6 \end{bmatrix} E_t \left[a_{t+1} \right] + \begin{bmatrix} \alpha_3 \\ 0 \end{bmatrix} a_t, \tag{7}$$

- (Conti.)

$$E_t \begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -\alpha_1 \frac{\alpha_4}{\alpha_5} & \frac{1}{\alpha_5} - \alpha_2 \frac{\alpha_4}{\alpha_5} \end{bmatrix} \begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\alpha_6}{\alpha_5} \end{bmatrix} E_t \left[a_{t+1} \right] + \begin{bmatrix} \alpha_3 \\ -\alpha_3 \frac{\alpha_4}{\alpha_5} \end{bmatrix} a_t .$$

The system can be written in a more compact matrix form:

$$E_t \begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = M \begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} + \zeta_t , \tag{8}$$

where

$$M = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -\alpha_1 \frac{\alpha_4}{\alpha_5} & \frac{1}{\alpha_5} - \alpha_2 \frac{\alpha_4}{\alpha_5} \end{bmatrix}, \zeta_t = \begin{bmatrix} 0 \\ -\frac{\alpha_6}{\alpha_5} \end{bmatrix} E_t \left[a_{t+1} \right] + \begin{bmatrix} \alpha_3 \\ -\alpha_3 \frac{\alpha_4}{\alpha_5} \end{bmatrix} a_t . \tag{9}$$

- (Conti.) Since the 2-by-2 matrix M is non-singular and has 2 linearly independent eigenvectors in our model, M can be diagonalizable as follows:

$$M = Q\Lambda Q^{-1}, \tag{10}$$

where Q has the eigenvectors of M as its columns, and Λ has eigenvalues of M down its diagonal:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{11}$$

where

$$\lambda_1 + \lambda_1 = \alpha_1 + \frac{1}{\alpha_5} - \alpha_2 \frac{\alpha_4}{\alpha_5} \text{ and } \lambda_1 \lambda_2 = \frac{\alpha_1}{\alpha_5}.$$

Using the parameter values we calibrated to the U.S. economy, we know that $0 < \lambda_1 < 1$ and $\lambda_2 > 1$.

- (Conti.) In MATLAB, `[V,D] = eig(X)` produces a diagonal matrix D of eigenvalues and a full matrix V whose columns are the corresponding eigenvectors so that X*V = V*D.
- For example, given

$$X = \begin{bmatrix} 1.075 & 0.25 \\ 0 & 0.9407 \end{bmatrix},$$

running `[V,D] =eig(X)` gives

$$V = \begin{bmatrix} 1 & -0.8810 \\ 0 & 0.4731 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1.075 & 0 \\ 0 & 0.9407 \end{bmatrix}.$$

- (Conti.) The solution to system (8) can be written

$$E_t\left(Q^{-1}\begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix}\right) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\left(Q^{-1}\begin{bmatrix} k_t \\ \lambda_t \end{bmatrix}\right) + Q^{-1}\zeta_t,$$

$$\text{or } E_t\left(\begin{bmatrix} \widetilde{k}_{t+1} \\ \widetilde{\lambda}_{t+1} \end{bmatrix}\right) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \widetilde{k}_t \\ \widetilde{\lambda}_t \end{bmatrix} + \widetilde{\zeta}_t, \quad (12)$$

In this case we end up with two scalar stochastic difference equations

$$E_t\left[\widetilde{k}_{t+1}\right] = \lambda_1 \widetilde{k}_t + \widetilde{\zeta}_{1,t}, \quad (13)$$

$$E_t\left[\widetilde{\lambda}_{t+1}\right] = \lambda_2 \widetilde{\lambda}_t + \widetilde{\zeta}_{2,t}. \quad (14)$$

We now would like to find solutions for $\left(\widetilde{k}_t, \widetilde{\lambda}_t\right)$ that are: (i) consistent with the predetermined variable k_{t+1} being determined at time t and (ii) $\left(\widetilde{k}_t, \widetilde{\lambda}_t\right)$ does not explode.

- (Conti.) Note that $\widetilde{\lambda}_t$ is a jump variable and does not have the initial condition, and $\lambda_2 > 1$. We thus solve (14) forward as follows

$$\begin{aligned} \widetilde{\lambda}_t &= -\sum_{j=0}^{\infty} (\lambda_2)^{-(j+1)} E_t\left[\widetilde{\zeta}_{2,t+j}\right] \qquad (15) \\ &= -\sum_{j=0}^{\infty} (\lambda_2)^{-(j+1)} \left[\begin{array}{cc} q^{21} & q^{22} \end{array}\right] E_t\left[\zeta_{t+j}\right], \end{aligned}$$

where we denote $Q^{-1} = \left[\begin{array}{cc} q^{11} & q^{12} \\ q^{21} & q^{22} \end{array}\right]$. Since $\widetilde{\lambda}_t = q^{21}k_t + q^{22}\lambda_t$, we have

$$\lambda_t = -\left(q^{22}\right)^{-1} q^{21}k_t + \left(q^{22}\right)^{-1} \widetilde{\lambda}_t. \qquad (16)$$

Combining (15) with (16) yields

$$\lambda_t = -\left(q^{22}\right)^{-1} q^{21}k_t - \left(q^{22}\right)^{-1} \sum_{j=0}^{\infty} (\lambda_2)^{-(j+1)} \left[\begin{array}{cc} q^{21} & q^{22} \end{array}\right] E_t\left[\zeta_{t+j}\right]. \qquad (17)$$

- (Conti.) Note that the original difference equation for k, $0 = -k_{t+1} + \alpha_1 k_t + \alpha_2 \lambda_t + \alpha_3 a_t$, can be written as

$$k_{t+1} = m_{11} k_t + m_{12} \lambda_t + \zeta_{1,t}. \tag{18}$$

Substituting (17) into this gives

$$\begin{aligned} k_{t+1} &= m_{11} k_t - m_{12} \left\{ \frac{q^{21}}{q^{22}} k_t + \frac{1}{q^{22}} \sum_{j=0}^{\infty} \lambda_2^{-(j+1)} \begin{bmatrix} q^{21} & q^{22} \end{bmatrix} E_t \left[\zeta_{t+j} \right. \right. \\ &\quad + \zeta_{1,t}, \\ &= \left[m_{11} - \frac{q^{21} m_{12}}{q^{22}} \right] k_t - \frac{m_{12}}{q^{22}} \sum_{j=0}^{\infty} \lambda_2^{-(j+1)} \begin{bmatrix} q^{21} & q^{22} \end{bmatrix} E_t \left[\zeta_{t+j} \right] \end{aligned}$$

- Since $a_{t+1} = \rho a_t + \varepsilon_{t+1}$,

$$\begin{aligned} \zeta_t &= \begin{bmatrix} 0 \\ -\frac{\alpha_6}{\alpha_5} \end{bmatrix} E_t \left[a_{t+1} \right] + \begin{bmatrix} \alpha_3 \\ -\alpha_3 \frac{\alpha_4}{\alpha_5} \end{bmatrix} a_t \\ &= \begin{bmatrix} \alpha_3 \\ -\rho \frac{\alpha_6}{\alpha_5} - \alpha_3 \frac{\alpha_4}{\alpha_5} \end{bmatrix} a_t = \psi a_t, \text{ and } E_t \left[\zeta_{t+j} \right] = \psi \rho^j a_t. \end{aligned}$$

- (Conti.) Substituting these into (17) into this gives

$$\begin{aligned}
\lambda_t &= -\frac{q^{21}}{q^{22}}k_t - \left(q^{22}\right)^{-1} \sum_{j=0}^{\infty}\left[\left(\lambda_2\right)^{-(j+1)} \left[\begin{array}{cc} q^{21} & q^{22} \end{array}\right] \psi\rho^j a_t\right] \\
&= -\frac{q^{21}}{q^{22}}k_t - \frac{1}{\lambda_2 q^{22}} \left(\sum_{j=0}^{\infty}\left[\left(\lambda_2\right)^{-j}\rho^j\right]\right) \left[\begin{array}{cc} q^{21} & q^{22} \end{array}\right] \psi a_t \\
&= -\frac{q^{21}}{q^{22}}k_t - \frac{1}{\lambda_2 - \rho} \left[\begin{array}{cc} \frac{q^{21}}{q^{22}} & 1 \end{array}\right] \psi a_t. \qquad (19)
\end{aligned}$$

- (Conti.) Similarly,

$$
\begin{aligned}
k_{t+1} &= \left[m_{11} - m_{12}\left(q^{22}\right)^{-1} q^{21}\right] k_t \\
&\quad - m_{12}\left(q^{22}\right)^{-1} \sum_{j=0}^{\infty} (\lambda_2)^{-(j+1)} \left[\begin{array}{cc} q^{21} & q^{22} \end{array}\right] E_t\left[\zeta_{t+j}\right] + \zeta_{1,t}, \\
&= \left[m_{11} - m_{12}\left(q^{22}\right)^{-1} q^{21}\right] k_t - \frac{m_{12}}{\lambda_2 - \rho} \left[\begin{array}{cc} \frac{q^{21}}{q^{22}} & 1 \end{array}\right] \psi a_t + \zeta_{1,t}, \\
&= \left[m_{11} - m_{12}\left(q^{22}\right)^{-1} q^{21}\right] k_t \qquad (20) \\
&\quad - m_{12} \frac{1}{\lambda_2 - \rho} \left[\begin{array}{cc} \frac{q^{21}}{q^{22}} & 1 \end{array}\right] \psi a_t + \left[\begin{array}{cc} 1 & 0 \end{array}\right] \psi a_t .
\end{aligned}
$$

- Note that the approach we just used implicitly assumes that the number of non-predetermined variables must be the same as the number of explosive roots. If the number of explosive roots is smaller, the system will have multiple solutions. If the number of explosive roots is bigger, the system will have no stationary solution.
- The flipside of this conclusion is that the number of predetermined variables should be equal to the number of stable roots.

The Calibrated Example (Hansen's RBC Model)

- In the last lecture note, we have calibrated the model parameters and obtained the following coefficients in the log-linearized system:

$$
\begin{aligned}
\alpha_1 &= \frac{\overline{Y}}{\overline{K}} + (1-\delta) = 1.075, \alpha_2 = \frac{\overline{C}}{\overline{K}} + \frac{1-\alpha}{\alpha}\frac{\overline{Y}}{\overline{K}} = 0.25, \\
\alpha_3 &= \frac{\overline{Y}}{\alpha \overline{K}} = 0.278, \alpha_4 = 0, \\
\alpha_5 &= 1 + (1-\alpha)\frac{\overline{Y}}{\overline{RK}} = 1.063, \alpha_6 = \frac{\overline{Y}}{\overline{RK}} = 0.1.
\end{aligned}
$$

  Substituting these numerical coefficients into the expression for M:

$$
M = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -\alpha_1 \frac{\alpha_4}{\alpha_5} & \frac{1}{\alpha_5} - \alpha_2 \frac{\alpha_4}{\alpha_5} \end{bmatrix} = \begin{bmatrix} 1.075 & 0.25 \\ 0 & \frac{1}{1.063} \end{bmatrix}.
$$

- Using a simple Matlab command: `[Q Lambda]=eig(M)`, we can decompose $M = Q\Lambda Q^{-1}$.

- (Conti.)

$$
\begin{aligned}
M &= \begin{bmatrix} 1 & -0.8810 \\ 0 & 0.4731 \end{bmatrix} \begin{bmatrix} 1.075 & 0 \\ 0 & 0.9407 \end{bmatrix} \begin{bmatrix} 1.0 & 1.8622 \\ 0 & 2.1137 \end{bmatrix} \\
&= \begin{bmatrix} -0.8810 & 1 \\ 0.4731 & 0 \end{bmatrix} \begin{bmatrix} 0.9407 & 0 \\ 0 & 1.075 \end{bmatrix} \begin{bmatrix} 0 & 2.1137 \\ 1 & 1.8622 \end{bmatrix},
\end{aligned}
$$

where $\lambda_1 = 0.9407$ and $\lambda_2 = 1.075$.

- We can now compute that

$$-\frac{q^{21}}{q^{22}} = -\frac{1}{1.8622} = -0.54, \quad (21)$$

$$\frac{1}{\lambda_2 - \rho} \begin{bmatrix} \frac{q^{21}}{q^{22}} & 1 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ -\rho\frac{\alpha_6}{\alpha_5} - \alpha_3\frac{\alpha_4}{\alpha_5} \end{bmatrix} = 0.48 \Longrightarrow \quad (22)$$

$$\lambda_t = -\frac{q^{21}}{q^{22}} k_t - \frac{1}{\lambda_2 - \rho} \begin{bmatrix} \frac{q^{21}}{q^{22}} & 1 \end{bmatrix} \psi a_t = 0.54 k_t - 0.48 a_t, \quad (23)$$

which is exactly what we obtained from using the undetermined coefficients method.

Blanchard-Kahn Method (2)

- If we know the exogenous process of a_t, we can apply the decomposition method to the RBC model by another way:

$$
\begin{aligned}
a_{t+1} &= \rho a_t + \varepsilon_{t+1}, && (24) \\
0 &= -k_{t+1} + \alpha_1 k_t + \alpha_2 \lambda_t + \alpha_3 a_t, && (25) \\
0 &= E_t \left[-\lambda_t + \alpha_4 k_{+1} + \alpha_5 \lambda_{t+1} + \alpha_6 a_{t+1} \right]. && (26)
\end{aligned}
$$

- The system can be written in a more compact matrix form

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_6 & \alpha_4 & \alpha_5 \end{bmatrix} E_t \begin{bmatrix} a_{t+1} \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ \alpha_3 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_t \\ k_t \\ \lambda_t \end{bmatrix}, \quad (27)
$$

$$
E_t \begin{bmatrix} a_{t+1} \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = M \begin{bmatrix} a_t \\ k_t \\ \lambda_t \end{bmatrix}. \quad (28)
$$

- (Conti.) Given that

$$
\begin{aligned}
M &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_6 & \alpha_4 & \alpha_5 \end{bmatrix}^{-1} \begin{bmatrix} \rho & 0 & 0 \\ \alpha_3 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \rho & 0 & 0 \\ \alpha_3 & \alpha_1 & \alpha_2 \\ -\frac{\rho\alpha_6}{\alpha_5} - \frac{\alpha_3\alpha_4}{\alpha_5} & -\frac{\alpha_1\alpha_4}{\alpha_5} & \frac{1}{\alpha_5} - \frac{\alpha_2\alpha_4}{\alpha_5} \end{bmatrix},
\end{aligned}
$$

it is straightforward to show that M can be diagonalizable as follows:

$$
M = Q\Lambda Q^{-1}, \tag{29}
$$

where Q has the eigenvectors of M as its columns, and Λ has eigenvectors of M down its diagonal:

$$
\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{30}
$$

- (Conti.) Therefore,

$$\begin{aligned} E_t\left(Q^{-1}\begin{bmatrix} a_{t+1} \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix}\right) &= \Lambda\left(Q^{-1}\begin{bmatrix} a_t \\ k_t \\ \lambda_t \end{bmatrix}\right), \text{or} \\ E_t\left(\widetilde{x}_{t+1}\right) &= \Lambda\widetilde{x}_t \end{aligned}$$

- To rule out the explosive solution, we set $\widetilde{x}_{i,t} = 0$ for all the $\widetilde{x}_{i,t}$ associated with an eigenvalue greater than 1. In our parameterized RBC model, we set

$$\begin{bmatrix} p^{21} & p^{22} & p^{23} \end{bmatrix}\begin{bmatrix} a_t \\ k_t \\ \lambda_t \end{bmatrix} = 0, \tag{31}$$

where $\begin{bmatrix} p^{21} & p^{22} & p^{23} \end{bmatrix}$ is the second row of Q^{-1}.

- (Conti.) (31) can be rewritten as

$$\lambda_t = -\frac{p^{22}}{p^{23}} k_t - \frac{p^{21}}{p^{23}} a_t, \text{ or } c_t = \frac{p^{22}}{p^{23}} k_t + \frac{p^{21}}{p^{23}} a_t. \tag{32}$$

Linear Quadratic Dynamic Programming

- Linear quadratic (LQ): The objective function is quadratic and the transition equation (function) is linear.
- This specification leads to the widely used optimal linear regulator problem, for which the Bellman equation can be solved quickly using linear algebra.
- We focus on the special case in which both the objective function and the transition function are both time invariant.
- LQ has two uses here:
  1. The dynamic decision problems arising from linear rational expectations models naturally take the form of an optimal linear regulator.
  2. It can be used to approximate non-linear quadratic models.

The Optimal Linear Regulator Problem

- The undiscounted optimal linear regulator problem is to maximize over choice of

$$\max_{\{u_t\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty} \left(x_t' R x_t + u_t' Q u_t \right), \tag{33}$$

subject to

$$x_{t+1} = A x_t + B u_t, \tag{34}$$

where x_0 given.

- Here x_t is a $(n \times 1)$ vector of state variables, u_t is a $(k \times 1)$ vector of control variables, R is a positive semidefinite symmetric matrix, Q is a positive definite symmetric matrix, A is a $(n \times n)$ matrix, B is a $(n \times k)$ matrix,
- We guess that the value function is

$$v(x) = -x' P x, \tag{35}$$

where P is a positive semidefinite symmetric matrix.

- Using the transition equation to eliminate next period's state, the Bellman equation can be written as

$$-x_t' P x_t = \max_{u_t} \left\{ -\left(x_t' R x_t + u_t' Q u_t\right) - \left(A x_t + B u_t\right)' P \left(A x_t + B u_t\right) \right\}. \tag{36a}$$

The FOC is

$$\left(Q + B'PB\right) u = -B'PAx \Rightarrow \tag{37}$$

$$\text{The policy function:} \quad u = -Fx \triangleq -\left(Q + B'PB\right)^{-1} B'PAx.$$

Here we use the following matrix formulas:

$$\begin{aligned} \frac{\partial \left(a'x\right)}{\partial x} &= a, \frac{\partial \left(x'Ax\right)}{\partial x} = \left(A + A'\right) x, \frac{\partial \left(x'Ay\right)}{\partial x} = Ay, \\ \frac{\partial \left(x'Ay\right)}{\partial y} &= A'x, \frac{\partial \left(x'Ax\right)}{\partial A} = xx'. \end{aligned}$$

- Substituting (37) into the Bellman equation, (36a), and rearranging gives:

$$P = R + A'PA - A'PB\left(Q + B'PB\right)^{-1}B'PA. \tag{38}$$

This equation is called *the algebraic Riccati equation*. It expresses P as an implicit function of R, Q, A, B.

- Under some conditions, (38) has a unique positive semidefinite solution, which is approached in the limit as $j \to \infty$ by iterations on the following matrix Riccati difference equation:

$$P_{j+1} = R + A'P_jA - A'P_jB\left(Q + B'P_jB\right)^{-1}B'P_jA, \tag{39}$$

starting from $P_0 = 0$. The policy function associated with P_j is

$$F_{j+1} = \left(Q + B'P_jB\right)^{-1}B'P_jA. \tag{40}$$

Discounted Linear Regulator Problem

- The discounted optimal linear regulator problem is to maximize over choice of
$$\max_{\{u_t\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty}\left[\beta^t\left(x_t' R x_t + u_t' Q u_t\right)\right]. \tag{41}$$
-

$$P_{j+1} = R + \beta A' P_j A - \beta^2 A' P_j B\left(Q + B' P_j B\right)^{-1} B' P_j A, \tag{42}$$

starting from $P_0 = 0$. The optimal policy is

$$u = -\underbrace{\beta\left(Q + \beta B' P B\right)^{-1} B' P A}_{F} x, \tag{43}$$

Discounted Linear Regulator Problem with Cross Terms

- The discounted optimal linear regulator problem is to maximize over choice of

$$\max_{\{u_t\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty} \left[\beta^t \left(x_t' R x_t + u_t' Q u_t + 2 x_t' W u_t \right) \right]. \tag{44}$$

- Exercise to show that:

$$P_{j+1} = R + \beta A' P_j A - \left(\beta A' P_j B + W' \right) \left(Q + B' P_j B \right)^{-1} \left(\beta B' P_j A + W \right), \tag{45}$$

starting from $P_0 = 0$. The optimal policy is

$$u = -\underbrace{\left(Q + \beta B' P B \right)^{-1} \left(\beta B' P A + W \right)}_{F} x, \tag{46}$$

The Stochastic Optimal Linear Regulator Problem

- The stochastic LQ problem is to maximize over choice of

$$\max_{\{u_t\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty} E_0\left[\beta^t\left(x_t'Rx_t+u_t'Qu_t\right)\right], \tag{47}$$

$$s.t. x_{t+1}=Ax_t+Bu_t+C\epsilon_{t+1}, \tag{48}$$

where x_0 given, and ϵ_{t+1} is an $(n\times 1)$ vector of random variables that is iid according to the normal distribution with mean vector zero and covariance matrix $E\left[\epsilon_{t+1}\epsilon_{t+1}'\right]=I$. The assumptions on the matrices, R, Q, A, B, are the same as before.

- The value function is

$$v(x)=-x'Px-d, \tag{49}$$

where P is a positive semidefinite solution of the discounted Riccati equation corresponding to (42). As before, it is the limit of iterations on (42) starting from $P_0=0$.

- The scalar d is

$$d = \beta \left(1 - \beta\right)^{-1} \text{trace}\left(PCC'\right). \tag{50}$$

- The optimal policy continues to be where

$$u = -\underbrace{\beta \left(Q + \beta B'PB\right)^{-1} B'PA}_{F} x. \tag{51}$$

Definition

Certainty Equivalence Principle: The policy function that solves the stochastic optimal linear regulator problem is identical with the function for the corresponding nonstochastic linear optimal regulator problem.

Proof.

Substituting the guessed value function (49) into the Bellman equation gives

$$-x'Px - d = \max_u \left\{ \begin{array}{c} -\left(x'Rx + u'Qu\right) \\ -\beta E\left[\left(Ax + Bu + C\epsilon\right)' P \left(Ax + Bu + C\epsilon\right)\right] - \beta d \end{array} \right\},$$

where ϵ is the realization of ϵ_{t+1} when $x_t = x$ and where $E\left[\epsilon | x\right] = 0$. The preceding equation implies

$$\begin{aligned} & -x'Px - d \\ = \; & \max_u \left\{ \begin{array}{c} -\left(x'Rx + u'Qu\right) \\ -\beta E\left[\left(x'A' + u'B' + \epsilon'C'\right) P \left(Ax + Bu + C\epsilon\right)\right] - \beta d \end{array} \right\} \\ = \; & \max_u \left\{ \begin{array}{c} x'Rx + u'Qu - \beta d \\ -\beta E\left[\left(\begin{array}{c} x'A'PAx + x'A'Bu + x'A'PC\epsilon + u'B'PAx + u'B'PBu \\ +u'B'PC\epsilon + \epsilon'C'PAx + \epsilon'C'PBu + \epsilon'C'PC\epsilon \end{array} \right.\right. \end{array} \right. \end{aligned}$$

□

Proof.

(conti.) Using the fact that $E[\epsilon|x] = 0$,

$$\begin{aligned} & -x'Px - d \\ = & \max_u \left\{ \begin{array}{c} -(x'Rx + u'Qu) - \beta d \\ -\beta E[(x'A'PAx + (x'A'Bu + u'B'PAx) + u'B'PBu + \epsilon'C'PC\epsilon \end{array} \right. \\ = & \max_u - \left\{ \begin{array}{c} (x'Rx + u'Qu) + \beta d \\ +\beta x'A'PAx + \beta 2x'A'Bu + \beta u'B'PBu + E[\epsilon'C'PC\epsilon] \end{array} \right\} \end{aligned}$$

The FOC is

$$\left(Q + \beta B'PB\right) u = -\beta B'PAx, \text{ i.e., } u = -\beta \left(Q + \beta B'PB\right)^{-1} B'PAx. \tag{52}$$

Using $E[\epsilon'C'PC\epsilon] = \operatorname{trace}(PCC')$,

$$P = R + \beta A'PA - \beta^2 A'PB\left(Q + B'PB\right)^{-1} B'PA, \tag{53}$$

$$d = \beta(1-\beta)^{-1} \operatorname{trace}(PCC'). \tag{54}$$

□

Stability

- Substituting $u = -Fx$ into the transition equation, $x_{t+1} = Ax_t + Bu_t$, gives
$$x_{t+1} = (A - BF)\, x_t.$$
This difference equation governs the evaluation of x_t under the optimal control.
- This system is said to be *stable* if $\lim_{t\to\infty} x_t = 0$ starting from any x_0. Assume that the eigenvalues of $A - BF$ are distinct, and use the eigenvalue decomposition: $A - BF = D\Lambda D^{-1}$, where the columns of D are the eigenvectors of $A - BF$ and Λ is a diagonal matrix of eigenvalues of $A - BF$. We can then write
$$x_{t+1} = D\Lambda D^{-1} x_t = D\Lambda^t D^{-1} x_0, \tag{55}$$
which means that the system is stable for any x_0 iff the eigenvalues of $A - BF$ are all strictly less than 1 in absolute value. If this condition is satisfied, $A - BF$ is called a stable matrix. Since F depends on R, Q, A, B, whether the system is stable or not depends on them.

- Assumption 1: The matrix R is positive definite.
- Proposition: Under assumption 1, if a solution of the undiscounted regulator exists, it satisfies $\lim_{t\to\infty} x_t = 0$. Note that if $x_t \nrightarrow 0$, then $\sum_{t=0}^{\infty} \left(-x_t' R x_t\right) \to -\infty$.

Definition

The pair (A, B) is said to be *stabilizable* if there exists a matrix F for which $A - BF$ is a stable matrix.

Theorem

If (A, B) is stabilizable and R is positive definite, then under the optimal rule F, $A - BF$ is a stable matrix.

Linear-Quadratic Approximation

- An important use of the optimal linear regulator is to approximate the solution of more complicated dynamic programs:

$$\max_{\{u_t\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \left[\beta^t r\left(z_t\right) \right] \right\}, \tag{56}$$

$$s.t. x_{t+1} = Ax_t + Bu_t + C\epsilon_{t+1}, \tag{57}$$

  where $\{\epsilon_{t+1}\}$ is a vector of iid random disturbances with mean 0 and finite variance, and $r\left(z_t\right)$ is a concave and twice continuously differentiable function of $z_t \equiv \left[\begin{array}{ccc} 1 & x_t & u_t \end{array}\right]'$.
- All nonlinearities in (56) are absorbed into the composite function $r\left(z_t\right)$. We need to replace by a quadratic $z'Mz$. We choose a point $\bar{z}$ and approximate with the first two moments of a Taylor series:

$$r\left(z\right) = F\left(x_t, u_t\right) \simeq z'Mz = \left[\begin{array}{cc} 1 & x_t' \end{array}\right] R \left[\begin{array}{c} 1 \\ x_t \end{array}\right] + u_t'Qu_t + 2u_t'W \left[\begin{array}{c} 1 \\ x_t \end{array}\right] \tag{58}$$

  where z is $(1+n+k) \times 1$ if x is $n \times 1$ and u is $k \times 1$.

- Note that the second-order Taylor expansion of $F(x_t, u_t)$ can be written as

$$F(x_t, u_t) = F(\overline{x}, \overline{u}) + \begin{bmatrix} F_x(\overline{x}, \overline{u})' & F_u(\overline{x}, \overline{u})' \end{bmatrix} \begin{bmatrix} (x_t - \overline{x}) \\ (u_t - \overline{u}) \end{bmatrix} +$$

$$\begin{bmatrix} (x_t - \overline{x})' & (u_t - \overline{u})' \end{bmatrix} \begin{bmatrix} \frac{1}{2}F_{xx}(\overline{x}, \overline{u}) & \frac{1}{2}F_{xu}(\overline{x}, \overline{u}) \\ \frac{1}{2}F_{ux}(\overline{x}, \overline{u}) & \frac{1}{2}F_{uu}(\overline{x}, \overline{u}) \end{bmatrix} \begin{bmatrix} (x_t - \overline{x}) \\ (u_t - \overline{u}) \end{bmatrix}$$

- Consider the $(1+n+k)\times(1+n+k)$ matrix

$$M=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \triangleq \begin{bmatrix} R & W \\ W' & Q \end{bmatrix}, \tag{59}$$

  where m_{11} is 1×1, m_{22} is $n\times n$, m_{33} is $k\times k$, and the rest of the matrices conform to make M square.
- The product

$$\begin{aligned} z'Mz &= m_{11}+\left(m_{12}+m_{21}'\right)x+\left(m_{13}+m_{31}'\right)u \\ &\quad +x'm_{22}x+x'\left(m_{23}+m_{32}'\right)u+u'm_{33}u, \end{aligned} \tag{60}$$

  where we have a constant term, two linear terms and the quadratic terms. By defining

$$\begin{aligned} m_{11} &= F\left(\overline{x},\overline{u}\right)-\overline{x}'F_x\left(\overline{x},\overline{u}\right)-\overline{u}'F_u\left(\overline{x},\overline{u}\right)+\frac{1}{2}\overline{x}'F_{xx}\left(\overline{x},\overline{u}\right)\overline{x}+ \\ &\quad \overline{x}'F_{xu}\left(\overline{x},\overline{u}\right)\overline{u}+\frac{1}{2}\overline{u}'F_{uu}\left(\overline{x},\overline{u}\right)\overline{u}, \end{aligned} \tag{61}$$

  all of *the constant terms* are included in m_{11}.

- Defining

$$m_{12} = m_{21}' = \frac{1}{2}\left(F_x\left(\overline{x},\overline{u}\right)' - \overline{x}' F_{xx}\left(\overline{x},\overline{u}\right) - \overline{u}' F_{ux}\left(\overline{x},\overline{u}\right)\right), \tag{62}$$

$$m_{13} = m_{31}' = \frac{1}{2}\left(F_u\left(\overline{x},\overline{u}\right)' - \overline{x}' F_{xu}\left(\overline{x},\overline{u}\right) - \overline{u}' F_{ux}\left(\overline{x},\overline{u}\right)\right) \tag{63}$$

all of *the linear terms* are included in M and make M a symmetric matrix.

- The *quadratic terms* are

$$m_{22} = \frac{1}{2} F_{xx}\left(\overline{x},\overline{u}\right), m_{23} = m_{32}' = \frac{1}{2} F_{xu}\left(\overline{x},\overline{u}\right), m_{33} = \frac{1}{2} F_{uu}\left(\overline{x},\overline{u}\right). \tag{64}$$

- The quadratic discounted dynamic programming is

$$\max_{\{u_t\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \left[\beta^t z_t' M z_t\right] \right\}, \tag{65}$$

$$s.t. \begin{bmatrix} 1 \\ x_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ x_t \end{bmatrix} + B u_t + C\epsilon_{t+1}, \tag{66}$$

where $z_t' \equiv \begin{bmatrix} 1 & x_t & u_t \end{bmatrix}$.

Example: Solving the Hansen RBC Model

- Consider the following RBC model:

$$\max_{\{C_t, K_{t+1}, L_t\}} E\left[\sum_{t=0}^{\infty} \beta^t \left(\ln C_t + B\left(1 - L_t\right)\right)\right], \tag{67}$$

$$s.t. Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = C_t + I_t, \tag{68}$$

$$K_{t+1} = (1-\delta) K_t + I_t, \tag{69}$$

$$A_{t+1} = \rho A_t + (1-\rho) + \varepsilon_{t+1}, \tag{70}$$

where the technology shock (A_t) is assumed to be an AR(1) process around the steady state $\overline{A} = 1$ and ε_{t+1} is an iid normal variable with mean 0 and variance ω^2.

- Substituting (68) and (69) into the objective function:

$$\max_{\{K_{t+1}, L_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{\ln\left[A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t - K_{t+1}\right] + B\left(1 - L_t\right)\right\} \tag{71}$$

- The budget constraints can be rewritten as

$$\begin{aligned} K_{t+1} &= K_{t+1} && (72) \\ \ln A_{t+1} &= \rho \ln A_t + \varepsilon_{t+1}, \text{or } a_{t+1} = \rho a_t + \varepsilon_{t+1}, && (73) \end{aligned}$$

where the state variables are K_t and a_t and the control variables are K_{t+1} and L_t:

$$\begin{aligned} \begin{bmatrix} 1 \\ K_{t+1} \\ a_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\rho & 0 & \rho \end{bmatrix} \begin{bmatrix} 1 \\ K_t \\ a_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_{t+1} \\ L_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \epsilon_{t+1}. && (74) \end{aligned}$$

- The second-order Taylor series expansion of (71):

$$u(\cdot) = \ln\left[\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha} - \delta\overline{K}\right] + A\left(1-\overline{L}\right) \tag{75}$$

$$+\frac{1}{\overline{C}}\left[\alpha\frac{\overline{Y}}{\overline{K}} + (1-\delta)\right]\left(K_t - \overline{K}\right) + \frac{\overline{Y}}{\overline{C}}\left(A_t - \overline{A}\right)$$

$$-\frac{1}{\overline{C}}\left(K_{t+1} - \overline{K}\right) + \left[\frac{(1-\alpha)\overline{Y}}{\overline{C}\overline{L}} - \frac{B}{1-\overline{L}}\right]\left(L_t - \overline{L}\right) \tag{76}$$

$$+\begin{bmatrix} K_t - \overline{K} \\ A_t - \overline{A} \\ K_{t+1} - \overline{K} \\ L_t - \overline{L} \end{bmatrix}' \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} K_t - \overline{K} \\ A_t - \overline{A} \\ K_{t+1} - \overline{K} \\ L_t - \overline{L} \end{bmatrix}$$

- The constant elements of the matrix are $\ln\left[A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t - K_{t+1}\right]$

$$
\begin{aligned}
a_{11} &= -\frac{1}{2\overline{C}^2}\left[\alpha\frac{\overline{Y}}{\overline{K}} + (1-\delta)\right]^2 - \frac{1}{2\overline{C}}\alpha(1-\alpha)\frac{\overline{Y}}{\overline{K}^2}, \\
a_{12} &= a_{21} = -\frac{\overline{Y}}{\overline{C}^2}\left[\alpha\frac{\overline{Y}}{\overline{K}} + (1-\delta)\right] + \alpha\frac{1}{2\overline{C}}\frac{\overline{Y}}{\overline{K}}, \\
a_{13} &= a_{31} = \frac{1}{2\overline{C}^2}\left[\alpha\frac{\overline{Y}}{\overline{K}} + (1-\delta)\right], \\
a_{14} &= a_{41} = -\frac{1}{2\overline{C}^2}\left[\alpha\frac{\overline{Y}}{\overline{K}} + (1-\delta)\right]\frac{(1-\alpha)\overline{Y}}{\overline{L}} + \frac{1}{2\overline{C}}\alpha(1-\alpha)\frac{\overline{Y}}{\overline{K}\,\overline{L}}.
\end{aligned}
$$

Similarly, we can obtain other a_{ij}s. The matrix quadratic version of the nonlinear objective function is $z_t' M z_t$. The 5×5 matrix M can be written as

$$
M = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
m_{21} & a_{11} & a_{12} & a_{13} & a_{14} \\
m_{31} & a_{21} & a_{22} & a_{23} & a_{24} \\
m_{41} & a_{31} & a_{32} & a_{33} & a_{34} \\
m_{51} & a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}.
$$

-
$$\begin{aligned}
m_{11} &= \ln\left(\overline{A}\overline{K}^{\alpha}\overline{L}^{1-\alpha}-\delta\overline{K}\right)+A\left(1-\overline{L}\right)-\frac{1}{\overline{C}}\left[\alpha\frac{\overline{Y}}{\overline{K}}+(1-\delta)-1\right]\overline{K} \\
&\quad -\frac{\overline{Y}}{\overline{C}}\overline{A}-\left[\frac{(1-\alpha)\overline{Y}}{\overline{C}\overline{L}}-\frac{B}{1-\overline{L}}\right]\overline{L} \\
&\quad +\begin{bmatrix}\overline{K}\\ \overline{A}\\ \overline{K}\\ \overline{L}\end{bmatrix}'\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}\overline{K}\\ \overline{A}\\ \overline{K}\\ \overline{L}\end{bmatrix}. \\
m_{12} &= m_{21}=\frac{1}{\overline{C}}\left[\alpha\frac{\overline{Y}}{\overline{K}}+(1-\delta)\right]-\begin{bmatrix}\overline{K}\\ \overline{A}\\ \overline{K}\\ \overline{L}\end{bmatrix}'\begin{bmatrix}a_{11}\\ a_{21}\\ a_{31}\\ a_{41}\end{bmatrix}.
\end{aligned}$$

-

$$
\begin{aligned}
m_{13} &= m_{31} = \frac{\overline{Y}}{\overline{C}} - \begin{bmatrix} \overline{K} \\ \overline{A} \\ \overline{K} \\ \overline{L} \end{bmatrix}' \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}, \\
m_{14} &= m_{41} = -\frac{1}{\overline{C}} - \begin{bmatrix} \overline{K} \\ \overline{A} \\ \overline{K} \\ \overline{L} \end{bmatrix}' \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}, \\
m_{15} &= m_{51} = \left[\frac{(1-\alpha)\overline{Y}}{\overline{C}\,\overline{L}} - \frac{B}{1-\overline{L}} \right] - \begin{bmatrix} \overline{K} \\ \overline{A} \\ \overline{K} \\ \overline{L} \end{bmatrix}' \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
P_{j+1} &= R + \beta A' P_j A - \left(\beta A' P_j B + W'\right) \left(Q + B' P_j B\right)^{-1} \left(\beta B' P_j A + \right. \\
&\Rightarrow P \text{ and } u = -Fx = -\left(Q + \beta B' P B\right)^{-1} \left(\beta B' P A + W\right) x.
\end{aligned}
$$

Hansen's Model with Permanent Technology Shocks

- We now discuss how to solve Hansen's RBC model with permanent shocks. Consider the RBC model we discussed above but now assume that the technology shock is a random walk:

$$\ln A_{t+1} = \ln A_t + \varepsilon_{t+1}, \text{or } a_{t+1} = a_t + \varepsilon_{t+1}, \tag{77}$$

  where ε_{t+1} is called a *permanent* shock to technology. Given the length of the data, it is very difficult to distinguish the case when ρ is close to 1 and the case when $\rho = 1$.
- Note that in this case the model no longer has a steady state in terms of (C, K). However, we can use the same solution methods to solve the model by variable transformation:

$$X_t = \frac{C_t}{A_t}, U_t = \frac{K_t}{A_t}, V_t = \frac{Y_t}{A_t} \tag{78}$$

  The optimizing problem can be restated in terms of these quantities and the model has a steady state in terms of (X, U, V, L).

- (Conti.) Given

$$A = \frac{1}{C_t}(1-\alpha)\frac{Y_t}{L_t}, \tag{79}$$

$$\frac{1}{C_t} = \beta E_t\left[R_{t+1}\frac{1}{C_{t+1}}\right], \tag{80}$$

$$K_{t+1} = (1-\delta)K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t, \tag{81}$$

$$R_{t+1} = 1-\delta+\alpha\frac{Y_{t+1}}{K_{t+1}}, \tag{82}$$

we have the following system:

$$A = \frac{1}{X_t}(1-\alpha)\frac{V_t}{L_t}, \tag{83}$$

$$\frac{1}{X_t} = \beta E_t\left[R_{t+1}\frac{1}{X_{t+1}}\frac{A_t}{A_{t+1}}\right], \tag{84}$$

$$U_{t+1}\frac{A_{t+1}}{A_t} = (1-\delta)U_t + A_t U_t^{\alpha} L_t^{1-\alpha} - X_t, \tag{85}$$

$$R_{t+1} = 1-\delta+\alpha\frac{V_{t+1}}{U_{t+1}}, \tag{86}$$

- (Conti.) The new system can also be log-linearized around the steady state in terms of (X, U, V, L)

$$
\begin{aligned}
A &= \frac{1}{X_t}(1-\alpha)\frac{V_t}{L_t}, \Longrightarrow -x_t + v_t - l_t = 0, && (87) \\
\frac{1}{X_t} &= \beta E_t\left[R_{t+1}\frac{1}{X_{t+1}}\frac{A_t}{A_{t+1}}\right], \\
&\Longrightarrow -x_t = E_t\left[r_{t+1} - x_{t+1} + a_t - a_{t+1}\right] \\
&\Longrightarrow -x_t = E_t\left[r_{t+1} - x_{t+1}\right] && (88) \\
U_{t+1}\frac{A_{t+1}}{A_t} &= (1-\delta)U_t + U_t^{\alpha}(A_tL_t)^{1-\alpha} - X_t, \\
&\Longrightarrow \overline{U}u_{t+1} + (a_{t+1} - a_t) = (1-\delta)\overline{U}u_t + \overline{V}v_t - \overline{X}x_t, \\
&\Longrightarrow \overline{U}u_{t+1} + \varepsilon_{t+1} = (1-\delta)\overline{U}u_t + \overline{V}v_t - \overline{X}x_t, && (89) \\
v_t &= \alpha u_t + (1-\alpha) l_t, && (90) \\
R_{t+1} &= 1 - \delta + \alpha\frac{V_{t+1}}{U_{t+1}}, \Longrightarrow \overline{R}r_{t+1} = \alpha\frac{\overline{V}}{\overline{U}}(v_{t+1} - u_{t+1}) && (91)
\end{aligned}
$$

- (conti.) Solving the log-linearized system yields the following decision rule:

$$x_t = \eta_x u_t,\ l_t = \eta_l u_t,\ \ u_{t+1} = \eta_u u_t + \eta_a \varepsilon_{t+1},$$

By definition, we can recover the time series of original variables:

$$\begin{aligned} c_t &= a_t + \eta_x (k_t - a_t),\ l_t = \eta_l (k_t - a_t), \\ k_{t+1} &= a_{t+1} + \eta_u (k_t - a_t) + \eta_a (a_{t+1} - a_t). \end{aligned}$$

Example: Using MATLAB to Simulate the Exogenous Process

```
clear
w=0;
A=0.95;
p=1;
C=0.0072;
n=1000;
ndisc=700;
v=arsim(w,A,C,n,ndisc); % using function "arsim"
plot(v(701:1000))
xlabel('Period, t')
ylabel('ln(A_t)')
title('An AR(1) process with \rho=0.95 and \omega=0.007')
```

Summary: Calibrate and Solve Nonlinear Stochastic Dynamic Optimization Models

1. Distinguish state variables (endogenous and exogenous) and control variables.
2. Derive all the first-order conditions using optimal control or dynamic programming (will be introduced in the next lecture).
3. Determine the steady state of the model economy.
4. Using the long-run evidence and the steady state conditions to pin down the values of model parameters.
5. Linearize or log-linearized the nonlinear dynamic system around its steady state.
6. Using the solution methods (method of undetermined coefficients, matrix decomposition, or LQ regulator) to solve the decision rules mapping state variables to control and other flow variables. (Note that solving two second-order difference equation is equivalent to solving the system of first-order difference equations.)