Lecture 7: Intertemporal Consumption and Saving

Yulei Luo

Economics, HKU

November 22, 2017

The Importance of Consumption

- Consumption is important to both economic growth and business cycles. In the U.S., consumption accounts for more than 70% in real GDP.
- How consumers/households allocate their incomes between consumption and saving given constraints determines how much to consume and how much to invest in the economy and thus affects standards of living in the long-run.
- With regard to fluctuations, consumption makes up the significant component of the demand for final goods and services, so understanding consumption is central to understanding how fiscal and monetary policies affect aggregate demand and total production.

Two Basic Assumptions in the Intertemporal Consumption Literature

1. Economic agents *maximize* an *intertemporal* utility function, defined on consumption levels in each period of the optimization horizon, subject to the resource/budget constraint.
2. Under uncertainty, the maximization is based on conditional expectations of future relevant variables (e.g., labor income, the interest rate) formed *rationally* by agents (i.e., the agents use all available information to do the best predictions on the future relevant variables).

Hall's Permanent Income Model

- Under these assumptions, the typical consumer's optimal consumption decision can be specified by the following problem:

$$v\left(w_t\right)=\max_{\left\{c_{t+i},w_{t+1+i}\right\}} E_t\left\{\sum_{i=0}^{\infty}\beta^i\left[-\frac{1}{2}\left(\overline{c}-c_{t+i}\right)^2\right]\right\} \tag{1}$$

subject to the constraint (for $i=0,\cdots,\infty$):

$$w_{t+1+i}=Rw_{t+i}-c_{t+i}+y_{t+i}, \tag{2}$$

w_t given. Here $\beta=\frac{1}{1+\rho}$ is the discount factor and ρ is the discount rate. $R=1+r$ is the gross interest rate.

- $\overline{c}$ is the bliss point, and can be linked to the coefficient of relative risk aversion (CRRA):

$$\gamma=-\frac{cu''\left(c\right)}{u'\left(c\right)}=\frac{c}{\overline{c}-c}. \tag{3}$$

- Here $E_t[\cdot]$ is the rational expectation formed using all available information at t: For a random variable x, $E_t[x_{t+i}] = E_t[x_{t+i}|I_t]$. The RE hypothesis implies:

$$E_t[x_{t+i} - E_t[x_{t+i}]] = 0, \tag{4}$$

i.e., the forecast error, $x_{t+i} - E_t[x_{t+i}]$, is uncorrelated with the variables in the time-t information set, I_t. Note that in the above model, the values of y_t, c_t, and w_t are all included in I_t.

- The no-Ponzi-game (nPg) condition is satisfied:

$$\lim_{j\to\infty}\left(\frac{w_{t+j}}{R^j}\right) \geq 0. \tag{5}$$

Note that without imposing such a condition, interests on debt could be paid for by further borrowing on an infinite horizon. In addition, (2) must be binding

$$\lim_{j\to\infty}\left(\frac{w_{t+j}}{R^j}\right) = 0. \tag{6}$$

because the marginal utility of consumption is always positive.

Intertemporal Budget Constraint

- From (2) at t, repeatedly substituting w_{t+i} up to $t+j$, we obtain

$$\frac{1}{R}\sum_{i=0}^{j-1}\left(\frac{1}{R}\right)^{i}c_{t+i}+\left(\frac{1}{R}\right)^{j}w_{t+j}=\frac{1}{R}\sum_{i=0}^{j-1}\left(\frac{1}{R}\right)^{i}y_{t+i}+w_{t}. \tag{7}$$

- When the horizon j goes to infinity,

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}c_{t+i}+\lim_{j\to\infty}\left(\frac{1}{R}\right)^{j}w_{t+j}=\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}y_{t+i}+w_{t}. \tag{8}$$

  Using (6), we obtain the following *intertemporal budget constraint (IBC)* at the beginning of t:

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}c_{t+i}=\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}y_{t+i}+w_{t}. \tag{9}$$

Optimal Conditions

- Setting up the Lagrangian as

$$L = E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left[-\frac{1}{2} \left(\overline{c} - c_{t+i} \right)^2 \right] + \lambda \left[\begin{array}{c} \frac{1}{R} \sum_{i=0}^{\infty} \left(\frac{1}{R} \right)^i y_{t+i} \\ +w_t - \frac{1}{R} \sum_{i=0}^{\infty} \left(\frac{1}{R} \right)^i c_{t+i} \end{array} \right] \right\}, \tag{10}$$

  where λ is the Lagrange multiplier associated with the IBC.

- The Euler equation is thus

$$E_t \left[u' \left(c_{t+i} \right) \right] = \beta R E_{t+1} \left[u' \left(c_{t+1+i} \right) \right] \tag{11}$$

  for $i = 0, 1, 2, \cdots$. Note that in the first period $(i = 0)$, we have

$$u' \left(c_t \right) = \beta R E_t \left[u' \left(c_{t+1} \right) \right] = \frac{1+r}{1+\rho} E_t \left[u' \left(c_{t+1} \right) \right]. \tag{12}$$

- Note that the same reasoning applies to *any period t* when the optimization problem is solved: (12) gives the dynamics of the marginal utility in any two successive periods.

Economic Implications

- Note that under uncertainty, the expected value of utility may differ from its realization. Letting

$$u'(c_{t+1}) - E_t\left[u'(c_{t+1})\right] \equiv \eta_{t+1}, \tag{13}$$

we have

$$E_t\left[\eta_{t+1}\right] = 0$$

under the RE hypothesis. Consequently,

$$u'(c_{t+1}) = \frac{1+\rho}{1+r}u'(c_t) + \eta_{t+1}. \tag{14}$$

- When $r = \rho$,

$$u'(c_{t+1}) = u'(c_t) + \eta_{t+1}, \tag{15}$$

which means that the change of marginal utility from t to $t+1$ is given by a unforecastable stochastic term, $E_t\left[\eta_{t+1}\right] = 0$.

Consumption Dynamics

- Give the specified utility function, we have

$$c_{t+1} = c_t + u_{t+1}, \tag{16}$$

  where $u_{t+1} = -\eta_{t+1}$ and $E_t [u_{t+1}] = 0$.
- Therefore, consumption should be a martingale:

$$E_t [c_{t+1}] = c_t. \tag{17}$$

  This is the main implication of the intertemporal consumption model with quadratic utility and $\beta R = 1$. It means that
  - The best forecast of next period's consumption is current consumption.
  - The consumption change from t to $t+1$ cannot be forecast on the available information at t.
  - u_{t+1} is orthogonal to the time$-t$ information set used to form the conditional expectation (RE) $E_t [\cdot]$.

- Combining the consumption dynamics and the IBC, we can obtain the consumption function linking the state variables (w, y) to the control variable (c). Specifically, since the *realizations* of income and consumption must satisfy the IBC, (9), it should also hold with expectations: Applying $E_t[\cdot]$ on both sides of (9) gives

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i E_t[c_{t+i}] = \frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i E_t[y_{t+i}] + w_t. \tag{18}$$

- Substituting $E_t[c_{t+i}]$ with c_t on this equation yields:

$$\frac{1}{R-1}c_t = w_t + \frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i E_t[y_{t+i}] \triangleq w_t + h_t,$$

which can be rewritten as

$$c_t = (R-1)\left(w_t + \frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i E_t[y_{t+i}]\right) = r(w_t + h_t) \triangleq y_t^p, \tag{19}$$

- Note that h_t above, the present value at t of current and future expected labor income, can be regarded as the consumer's human wealth.
- (19) clearly shows that optimal consumption is determined by the level of financial wealth (w_t) and human wealth (h_t). The total wealth of the consumer at the beginning of t is given by $w_t + h_t$, and consumption at t is then the annuity value of total wealth, i.e., the wealth's return in each period: $r(w_t + h_t)$. This return is called *permanent income* measuring the flow that can be earned forever from the stock of total wealth.
- The conclusion is: the consumer chooses to consume in each period exactly his or her permanent income, computed based on conditional expectations of current and future labor income.

Permanent Income and Consumption

- Given (19), we know that the evolution of consumption and permanent income are the same. Leading (19) for one period gives

$$y_{t+1}^{p} = r\left(w_{t+1} + h_{t+1}\right), \tag{20}$$

which means that

$$y_{t+1}^{p} - E_t\left[y_{t+1}^{p}\right] = r\left(h_{t+1} - E_t\left[h_{t+1}\right]\right), \tag{21}$$

because $w_{t+1} = E_t\left[w_{t+1}\right]$. (Note that $w_{t+1} = Rw_t - c_t + y_t$.)

- This expression shows that permanent income calculated at $t+1$, conditional on available information at that time, *differs* from its conditional expectations formed one period earlier only if there is a surprise in the consumer's human wealth at $t+1$.
- That is, the *surprise* in permanent income at $t+1$ is equal to *the annuity value* (r) of the *surprise* in human wealth due to new information on future labor income, available only at $t+1$.

- Using the fact that $E_t[c_{t+1}] = c_t$ and $c_t = y_t^p$,

$$E_t\left[y_{t+1}^p\right] = y_t^p, \tag{22}$$

i.e., permanent income y_t^p is the best forecast of the next period's permanent income, y_{t+1}^p.
- Using this expression, we have

$$\begin{aligned} y_{t+1}^p &= y_t^p + r\left(h_{t+1} - E_t[h_{t+1}]\right) \\ &= y_t^p + r\left(E_{t+1}[h_{t+1}] - E_t[h_{t+1}]\right) \\ &= y_t^p + r\left[\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i (E_{t+1} - E_t)\left[y_{t+1+i}\right]\right], \end{aligned} \tag{23}$$

which means that the surprise in h_{t+1} can be expressed as the revision in expectations on future income: y_t^p changes over time only if these expectations change, i.e., $(E_{t+1} - E_t)\left[y_{t+1+i}\right]$ is non-zero.

- Therefore,

$$
\begin{aligned}
c_{t+1} &= c_t + r\left[\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i (E_{t+1} - E_t)\left[y_{t+1+i}\right]\right] \\
&\triangleq c_t + u_{t+1},
\end{aligned}
\tag{24}
$$

which means that the change in consumption between t and $t+1$ is unpredictable as of time t since it only depends on the $t+1$ available information: $E_t\left[u_{t+1}\right] = 0$.

Implications for Saving

- To find the implications of the permanent income hypothesis for saving, we start with the definition of disposable income y^d:

$$y^d \equiv rw_t + y_t,$$

and saving is defined as the difference between disposable income and consumption:

$$s_t \equiv y^d - c_t = y^d - y_t^p = y_t - rh_t, \tag{25}$$

which means that time t saving is equal to the difference between current income (y_t) and the annuity value of human wealth (h_t).

- This difference does not affect consumption: If $y_t > rh_t$, it is entirely saved, whereas, if $y_t < rh_t$, it determines a de-accumulation of financial wealth of an equal amount.

- Note that saving can be rewritten as

$$
\begin{aligned}
s_t &= y_t - rh_t \\
&= y_t - (R-1)\left[\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i E_t\left[y_{t+i}\right]\right] \\
&= \frac{1}{R}y_t - \left[\frac{1}{R} - \left(\frac{1}{R}\right)^2\right] E_t\left[y_{t+1}\right] \\
&\quad - \left[\left(\frac{1}{R}\right)^2 - \left(\frac{1}{R}\right)^3\right] E_t\left[y_{t+1}\right] - \cdots \\
&= -\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^i E_t\left[\Delta y_{t+i}\right]. \qquad (26)
\end{aligned}
$$

- Economic interpretations: The consumer saves, accumulating financial wealth, to face expected future declines in labor income: a saving for a raining day motive.

Current Income, Permanent Income, and Consumption

- To explore the relation between current and permanent income we assume a simple AR(1) process generating labor income y:

$$y_{t+1} = \rho y_t + (1-\rho)\,\overline{y} + \varepsilon_{t+1}, \tag{27}$$

where ε_{t+1} is an iid innovation with mean 0 and variance ω^2, $\overline{y}$ is the unconditional mean of income, and $\rho \in [0,1]$. In addition, we assume that this income process is in the consumer's information set.

- Substituting (27) into (24), we can obtain the expectation revision:

$$(E_{t+1} - E_t)\left[y_{t+1+i}\right] = \rho^i \varepsilon_{t+1}, \tag{28}$$

for any $i \geq 0$, and

$$c_{t+1} = c_t + \frac{R-1}{R}\sum_{i=0}^{\infty}\left[\left(\frac{1}{R}\right)^i \rho^i \varepsilon_{t+1}\right] = c_t + \frac{R-1}{R-\rho}\varepsilon_{t+1}, \tag{29}$$

which directly links current income innovation (ε_{t+1}) to current consumption level (c_{t+1}).

- (29) means that if there is an unexpected increase in income at $t+1$, ε_{t+1}, the consumer will increase consumption at $t+1$ and expected consumption in all future periods by the annuity value of the increase in human wealth:

$$\frac{R-1}{R-\rho}\varepsilon_{t+1},$$

and save the additional increase in income that is not consumed,

$$\varepsilon_{t+1}-\frac{R-1}{R-\rho}\varepsilon_{t+1}=\frac{1-\rho}{R-\rho}\varepsilon_{t+1}. \tag{30}$$

Starting from the next period, the return on this amount of savings will increase disposable income and thus make the consumer keep the higher level of consumption over the entire future horizon.

- Similarly, substituting (27) into (19) yields the consumption function:

$$\begin{aligned} c_t &= (R-1)\left(w_t+\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i E_t\left[y_{t+i}\right]\right) \\ &= (R-1)\left(w_t+\frac{1}{R-\rho}y_t\right)+\frac{1-\rho}{R-\rho}\overline{y}. \end{aligned} \tag{31}$$

The Role of the Persistence of Labor Income

- Given R, ρ plays an important role in determining optimal consumption and saving. We now consider two special cases.
- Case 1 ($\rho = 0$). In this case, income is an iid process:

$$y_{t+1} = \overline{y} + \varepsilon_{t+1}, \tag{32}$$

  which means that the innovation in current income is *purely transitory* and does not affect the level of income in future periods. Consequently,

$$c_{t+1} = c_t + \frac{R-1}{R}\varepsilon_{t+1}. \tag{33}$$

- Case 1 ($\rho = 1$). In this case, income is a random walk:

$$y_{t+1} = y_t + \varepsilon_{t+1}, \tag{34}$$

  which means that the innovation in current income is *permanent*, causing an equal change in all future income. Consequently,

$$c_{t+1} = c_t + \varepsilon_{t+1}. \tag{35}$$

Some puzzles in the Consumption Literature

- The dynamic implications of the PI model have motivated many empirical studies on consumption.
- Hall (1978) tests the orthogonality of the error term in the Euler equation with respect to past information. If the PI theory is correct, no variable known at $t-1$ can explain the changes in consumption between $t-1$ and t. Formally, the test is carried out by evaluating the statistical significance of $t-1$ variables in the t Euler equation. For example,

$$\Delta c_t = \alpha \Delta y_{t-1} + e_t, \tag{36}$$

  where $\alpha = 0$ if the PI theory holds. Hall showed that the null hypothesis ($\alpha = 0$) cannot be rejected for several past macro variables including income.
- Further studies focus on two puzzling observations that cannot be predicted by the PI theory:
  - The consumption's excess sensitivity to current income changes;
  - Its excess smoothness to income innovations.

Excess Sensitivity of Consumption to Current Income

- Flavin (1981): Added

$$\Delta y_t = \mu + \rho \Delta y_{t-1} + \varepsilon_t, \tag{37}$$

  where ε_t is an iid innovation, to the empirical analysis. According to the PI model,

$$\Delta c_t = \theta \varepsilon_t,$$

  where θ measures the intensity of the PI effect. Consumption is excessively sensitive to current income if c_t reacts to changes in y_t by *more than* is justified by the change in permanent income, $\theta \varepsilon_t$.
- Empirically, the excess sensitivity hypothesis can be formalized by:

$$\Delta c_t = \beta \Delta y_t + \theta \varepsilon_t + v_t, \tag{38}$$

  where β (if positive) measures the over-reaction of consumption to the change in current income, and v_t captures the effect of time t information about permanent income, but is unrelated to current income changes.

- According to PI, only innovations (i.e., unpredictable changes) in income cause consumption changes: the $\theta\varepsilon_t$ term captures precisely this effect.
- Therefore, an estimated value of β greater than 0 means that consumption over reacts *anticipated changes in income*.
- Note that this test is linked to Hall's orthogonality test:

$$\Delta c_t = \beta\mu + \beta\rho\Delta y_{t-1} + (\theta + \beta)\,\varepsilon_t + v_t, \tag{39}$$

  which means that $\beta = 0$ implies that $\alpha = 0$ in (36). That is, if consumption is excessively sensitive to income $(\beta > 0)$, then $\alpha > 0$.
- Flavin estimated $\beta = 0.36$ using the U.S. quarterly data. A potential explanation for the puzzle (the liquidity constraints that limit the consumer's borrowing capacity): With binding liquidity constraints, an increase in future income, though perfectly anticipated, affects consumption only when it actually occurs.

Relative Volatility of Consumption to Income

- In many empirical studies, the following income process cannot be rejected:
$$\Delta y_t = \mu + \rho \Delta y_{t-1} + \varepsilon_t, \tag{40}$$
where $\rho \in (0, 1)$. This specification means that the income level is permanently affected by the innovation ε. Using the U.S. quarterly data, Campbell and Deaton (1989) found that $\rho = 0.44$ is a fairly good statistical description of the dynamics of aggregate income. Using this specification,
$$c_{t+1} = c_t + r \left[\frac{1}{R} \sum_{i=0}^{\infty} \left(\frac{1}{R} \right)^i (E_{t+1} - E_t) \left[y_{t+1+i} \right] \right] = c_t + \frac{R}{R - \rho} \varepsilon_{t+1}.$$

- (Conti.) Here is the procedure to compute: $\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i (E_{t+1}-E_t)\left[y_{t+1+i}\right]$. Let's use Δ to denote this summation:

$$
\begin{aligned}
\Delta &= \sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i (E_{t+1}-E_t)\left[y_{t+1+i}\right] \\
&= (E_{t+1}-E_t)\left[\Delta y_{t+1}+y_t\right]+\left(\frac{1}{R}\right)(E_{t+1}-E_t)\left[\Delta y_{t+2}+y_{t+1}\right]+ \\
&= \sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i (E_{t+1}-E_t)\left[\Delta y_{t+1+i}\right] \\
&\quad +\left(\frac{1}{R}\right)\left[(E_{t+1}-E_t)\, y_{t+1}+\left(\frac{1}{R}\right)(E_{t+1}-E_t)\left[y_{t+2}\right]+\cdots\right] \\
&= \frac{\varepsilon_{t+1}}{1-\rho/R}+\left(\frac{1}{R}\right)\Delta,
\end{aligned}
$$

which implies that $\Delta = \frac{1}{(1-\rho/R)(1-1/R)}$.

- In this case, the relative volatility of consumption growth to income innovation can be written as

$$\frac{\mathrm{sd}(\Delta c)}{\mathrm{sd}(\varepsilon)} = \frac{R}{R-\rho} = 1.77 > 1$$

given that $R = 1.01$ and $\rho = 0.44$. That is, the implied volatility of consumption growth would be 1.77 times larger than that of the income innovation. However, using the U.S. quarterly data, Campbell and Deaton (1989) and Deaton (1992) estimated that this ratio is only about 0.64. It is called *the excess smoothness puzzle* in the literature. (It is also also called the Deaton Puzzle in the literature.)
- Some potential explanations for this puzzle: Habit formation, delayed adjustment due to imperfect information, etc.

Some Remarks Quadratic Utility and Consumption

- The Quadratic utility and its simple characterization of consumption and saving behavior has played a central role in recent empirical research. However, it is an unappealing description of behavior toward risk: It implies increasing absolution risk aversion:

$$ARA \triangleq -\frac{u''(\cdot)}{u'(\cdot)} = \frac{1}{\overline{c} - c} > 0, \tag{41}$$

  which means that the agent would pay more to avoid a given risky bet as wealth increases.

- Given that $\beta R = 1$ and $u'(c_t) = E_t[u'(c_{t+1})]$,

$$\overline{c} - c_t = E_t[\overline{c} - c_{t+1}],$$

  i.e., the variance of consumption or income has no effect on expected marginal utility, and thus on effect on optimal consumption. This is called the *certainty equivalence* result. (The consumption function under uncertainty is the same as that under certainty.)

- However, most plausible utility functions are such that $u'''(\cdot) > 0$, i.e., the marginal utility is a convex function.
- In this case, an increase in uncertainty raises the expected marginal utility, $E_t[u'(c_{t+1})]$. To maintain the Euler equation, $u'(c_t) = E_t[u'(c_{t+1})]$, c_t must decrease compared to the expected future consumption. That is, uncertainty leads consumers to defer consumption, to be more *prudent*.

Caballero's Permanent Income Model with CARA Utility

- Consider the following utility function

$$u(c) = -\frac{1}{\alpha} \exp(-\alpha c), \tag{42}$$

where $\alpha > 0$ is the coefficient of absolute risk aversion.

- As in the Hall model, we assume that

$$y_{t+1} = \rho y_t + (1 - \rho) \overline{y} + \varepsilon_{t+1}, \tag{43}$$

where ε_{t+1} is an iid innovation with mean 0 and variance ω^2. And $\beta R = 1$.

Optimal Conditions

- The FOCs for this problem is given by

$$\exp\left(-\alpha c_t\right) = E_t\left[\exp\left(-\alpha c_{t+1}\right)\right]. \tag{44}$$

- To proceed, we guess that consumption evolves according to the following process:

$$c_{t+1} = c_t + K_t + v_{t+1}, \tag{45}$$

  where K_t is a deterministic term and v_{t+1} is the innovation in consumption ($E_t\left[v_{t+1}\right] = 0$). Note that both the sequence of K_t and the stochastic features of v_{t+1} must be determined so as to satisfy the Euler equation (44) and the IBC specified before.
- Substituting the guessed consumption process into (44) gives

$$\exp\left(\alpha K_t\right) = E_t\left[\exp\left(-\alpha v_{t+1}\right)\right] \Longrightarrow K_t = \frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha v_{t+1}\right)\right]\right) \tag{46}$$

- (44) means that the value of K_t depends on the features of the distribution of v, yet to be determined.
- Using the Jensen's inequality,

$$\ln\left(E_t\left[\cdot\right]\right) > E_t\left[\ln\left(\cdot\right)\right],$$

and the property that $E_t\left[v_{t+1}\right] = 0$, we have

$$K_t = \frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha v_{t+1}\right)\right]\right) > \frac{1}{\alpha}E_t\left[-\alpha v_{t+1}\right] = 0, \qquad (47)$$

which means that consumption increases over time because the change in consumption between t to $t+1$ is expected to equal $K_t > 0$. Note that in the Hall model, consumption growth has zero mean.

Consumption Function

- To obtain the consumption function and then determine the effect of the precautionary saving motive on consumption, we use (45) and the IBC,

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}E_t\left[c_{t+i}\right]=w_t+\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}E_t\left[y_{t+i}\right], \tag{48}$$

to eliminate $E_t\left[c_{t+i}\right]$ and then obtain

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^{i}c_t+\frac{1}{R}\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^{i}\left(\sum_{j=1}^{i}K_{t+j-1}\right)=w_t+h_t,\Rightarrow$$

$$c_t=r\left(w_t+h_t\right)-\frac{R-1}{R}\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^{i}\left(\sum_{j=1}^{i}K_{t+j-1}\right), \tag{49}$$

which means that consumption is made up of a component that is just the permanent income obtained in the Hall model, $r\left(w_t+h_t\right)$, less a term which depends on Ks and captures the effect of the precautionary saving motive on consumption.

- The last step is to derive the form of the stochastic term v and its relationship to the income innovation ε. Specifically, substituting $y_{t+i} = E_t[y_{t+i}] + (y_{t+i} - E_t[y_{t+i}])$ into

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i c_{t+i} = \frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i y_{t+i} + w_t, \tag{50}$$

gives

$$\frac{1}{R}\sum_{i=0}^{\infty}\left(\frac{1}{R}\right)^i c_{t+i} = w_t + h_t + \frac{1}{R}\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^i (y_{t+i} - E_t[y_{t+i}]). \tag{51}$$

Using (49) to substitute c_t and (45) to substitute c_{t+i} $(i > 0)$, we obtain

$$\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^i \left(\sum_{j=1}^{i} v_{t+j}\right) = \sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^i (y_{t+i} - E_t[y_{t+i}]). \tag{52}$$

- Using

$$y_{t+i} - E_t\left[y_{t+i}\right] = \sum_{k=0}^{i-1} \rho^k \varepsilon_{t+i-k},$$

we have

$$\sum_{i=1}^{\infty} \left(\frac{1}{R}\right)^i \left(\sum_{j=1}^{i} v_{t+j}\right) = \sum_{i=1}^{\infty} \left(\frac{1}{R}\right)^i \left(\sum_{k=0}^{i-1} \rho^k \varepsilon_{t+i-k}\right). \quad (53)$$

Developing the summations, collecting terms containing v and ε with the same time subscript and using the fact that v and ε are serially uncorrelated processes, we obtain

$$\sum_{i=1}^{\infty} \left(\frac{1}{R}\right)^i \left(v_{t+h} - \rho^{i-1} \varepsilon_{t+h}\right) = 0, \forall h \geq 1, \Longrightarrow \quad (54)$$

$$v_{t+h} = \frac{R-1}{R-\rho} \varepsilon_{t+h}, \quad (55)$$

at all times $t + h$.

- Given (55) and (46), it is clear that K_t is constant because εs are iid. The consumption dynamics can be thus written as

$$c_{t+1} = c_t + K + \frac{R-1}{R-\rho}\varepsilon_{t+1}. \tag{56}$$

- Similarly, the consumption function can be written as

$$\begin{aligned} c_t &= r\left(w_t + h_t\right) - \frac{R-1}{R}\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^i\left(\sum_{j=1}^{i}K\right) \\ &= r\left(w_t + h_t\right) - \frac{R-1}{R}\sum_{i=1}^{\infty}\left(\frac{1}{R}\right)^i(iK) \\ &= r\left(w_t + h_t\right) - \frac{R-1}{R}K\frac{R}{(R-1)^2} \\ &= r\left(w_t + h_t\right) - \frac{K}{r}. \end{aligned}$$

Determining K

- Since ε follows a normal distribution:

$$\varepsilon \sim N\left(0, \omega^2\right),$$

we have

$$\begin{aligned} K_t &= \frac{1}{\alpha} \ln\left(E_t\left[\exp\left(-\alpha\theta\varepsilon_{t+1}\right)\right]\right), \\ &= \frac{1}{2}\alpha\theta^2\omega^2, \end{aligned}$$

where $\theta = \frac{R-1}{R-\rho}$. Note that here we use the following probability property: If $x \sim N\left(E[x], \sigma^2\right)$, then $\exp(x)$ follows a log-normal distribution with mean

$$E\left[\exp(x)\right] = \exp\left(E[x] + \frac{1}{2}\sigma^2\right)$$

Explicit Consumption Function and Dynamics

- We can now obtain

$$c_{t+1} = c_t + \frac{1}{2}\alpha\theta^2\omega^2 + \theta\varepsilon_{t+1}. \tag{57}$$

and

$$c_t = r\left(w_t + h_t\right) - \frac{1}{2r}\alpha\theta^2\omega^2. \tag{58}$$

- Economic interpretations: The innovation variance ω^2 has a *positive* effect on the change in consumption and a *negative* effect on the level of consumption. Increases in the uncertainty about future labor income lead to larger changes of consumption from one period to the next and drops in the level of current consumption.

- It can partly rationalize the observed slow decumulation of wealth by old individuals because they still have the precautionary saving motive.
- In addition, as argued in Caballero (1990), if positive innovation in current income is related to higher uncertainty about future income, the excess smoothness puzzle may be explained because a higher uncertainty induces consumers to save more and may thus reduce the response of consumption to income innovations.

Non-expected Utility/Recursive Utility (Optional)

- In the standard expected utility specification, the *risk and time* preferences are closely related: risk aversion and intertemporal substitution. For example, in the CRRA utility case, the coefficient of relative risk aversion is γ and intertemporal substitution is governed by $1/\gamma$. Any way to break this coincidence of time and risk?
- Epstein and Zin (1989) propose a class of utility functions that allows each dimension to be modeled *separately*. Specifically, preferences are defined *recursively* over current (known) consumption and a certainty equivalent of the next period's total utility:

$$U_t = U(c_t, c_{t+1}, c_{t+2,} \cdots) = W(c_t, R_t(U_{t+1})), \tag{59}$$

where $R_t(U_{t+1}) \triangleq \mathrm{CE}_{t+1}$ denotes the certainty equivalence in terms of period-t consumption of the uncertain total utility in the future periods:

$$R_t(U_{t+1}) = G^{-1}\left(E_t\left[G\left(U_{t+1}\right)\right]\right), \tag{60}$$

where W and G are increasing and concave.

- Most models in the literature consider simple functional forms for :

$$\rho > 0: W(c,z) = \left[(1-\beta)c^{1-1/\rho} + \beta z^{1-1/\rho}\right]^{1/(1-1/\rho)}, \tag{61}$$

$$\gamma > 0: \; G(x) = \frac{x^{1-\gamma}-1}{1-\gamma}. \tag{62}$$

- Note that the following limits hold:

$$\lim_{\rho \to 1} W(c,z) = c^{1-\beta} z^{\beta}. \tag{63}$$

$$\lim_{\gamma \to 1} G(x) = \ln x. \tag{64}$$

Proof.

Proof of (63). Given that $z = \mathrm{CE}_{t+1}$, the recursive utility can be written as

$$U(c_t, \mathrm{CE}_{t+1}) = \left[(1-\beta)c_t^{1-1/\rho} + \beta\left(\mathrm{CE}_{t+1}\right)^{1-1/\rho}\right]^{\frac{1}{1-1/\rho}} \Rightarrow \tag{65}$$

$$\ln U(c_t, \mathrm{CE}_{t+1}) = \frac{\ln\left[(1-\beta)c_t^{1-1/\rho} + \beta\left(\mathrm{CE}_{t+1}\right)^{1-1/\rho}\right]}{1-1/\rho}. \tag{66}$$

Using the de L'Hopital's rule, we obtain

$$\begin{aligned}\lim_{1/\rho\to 1} \ln U(c_t, \mathrm{CE}_{t+1}) &= \frac{\left[\begin{array}{c} -(1-\beta)c_t^{1-1/\rho}\ln c_t \\ -\beta\left(\mathrm{CE}_{t+1}\right)^{1-1/\rho}\ln \mathrm{CE}_{t+1}\end{array}\right]_{1/\rho=1}}{-1\cdot\left[(1-\beta)c_t^{1-1/\rho} + \beta\left(\mathrm{CE}_{t+1}\right)^{1-1/\rho}\right]_{1/\rho=1}} \\ &= (1-\beta)\ln c_t + \beta\ln\left(\mathrm{CE}_{t+1}\right).\end{aligned}$$

That is, $\lim_{\rho\to 1} U(c_t, \mathrm{CE}_{t+1}) = c^{1-\beta}\left(\mathrm{CE}_{t+1}\right)^{\beta}$. □

- If $\gamma = 1/\rho$ or if consumption is deterministic, we have the usual standard time-separable expected utility setting with the discount factor β, IES ρ, and RRA γ.
- Note that (65) can be rewritten as follows:

$$U(c_t, CE_{t+1}) = \left[(1-\beta) c_t^{\frac{1-\gamma}{\theta}} + \beta \left(CE_{t+1} \right)^{\frac{1-\gamma}{\theta}} \right]^{\frac{\theta}{1-\gamma}}, \qquad (67)$$

where $\gamma, \rho > 0, \gamma \neq 1,\ \theta = \frac{1-\gamma}{1-1/\rho}$.

- If $\gamma = 1/\rho$, $\theta = \frac{1-\gamma}{1-1/\rho} = 1$ and recursive substitution to eliminate U_{t+j} gives

$$U_t = \left[(1-\beta) E_t \sum_{j=0}^{\infty} \beta^j c_{t+j}^{1-\gamma} \right]^{1/(1-\gamma)}, \qquad (68)$$

which is equivalent to the standard expected utility expression:

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\gamma}}{1-\gamma}. \qquad (69)$$

CRRA Specification of the CE Function

- Consider a CRRA specification for the certainty equivalent (CE). For a random variable U_{t+1}, the period-$t+1$ onward utility, the certainty equivalent $\mathrm{CE}_t(U_{t+1})$ $(= \mathrm{CE}_{\,t+1})$ is defined as follows:

$$
\begin{aligned}
\left[CE_t(U_{t+1})\right]^{1-\gamma} &= E_t\left[U_{t+1}^{1-\gamma}\right] \text{ when } \gamma > 0 \text{ and } \gamma \neq 1, && (70)\\
&\Longrightarrow CE_t(U_{t+1}) = \left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\
\log CE_t(U_{t+1}) &= E_t\left[\log U_{t+1}\right] \text{ when } \gamma = 1, && (71)\\
&\Longrightarrow CE_t(U_{t+1}) = \exp\left(E_t\left[\log U_{t+1}\right]\right)
\end{aligned}
$$

CARA Specification of the CE Function

- Weil (1993): Consider first an infinitely-lived consumer whose preferences over deterministic consumption stream exhibit a *constant elasticity of intertermporal substitution*

$$U_t = U(c_t, c_{t+1}, c_{t+2,} \cdots) = \left\{ (1-\beta) \sum_{s=0}^{\infty} \beta^s c_{t+s}^{1-1/\rho} \right\}^{1/(1-1/\rho)}, \tag{72}$$

where ρ denotes constant elasticity of intertermporal substitution, and $\beta \in (0,1)$ is the discount factor. These preferences can, equivalently, be represented recursively as

$$\begin{aligned} U_t &= U(c_t, c_{t+1}, c_{t+2,} \cdots) \triangleq W(c_t, U(c_{t+1}, c_{t+2,} \cdots)) \\ &= \left\{ (1-\beta) c_t^{1-1/\rho} + \beta \left[U(c_{t+1}, c_{t+2,} \cdots) \right]^{1-1/\rho} \right\}^{1/(1-1/\rho)} \end{aligned} \tag{73}$$

where W is an aggregator function. Note that so far we haven't specified his attitude towards atemproral risk yet. His risk aversion is characterized by a CRRA under the standard time-additive EU.

- Weil (1993) assume that the behavior towards risk is summarized by *a constant coefficient of absolute risk aversion*, denoted by the parameter α:

$$\exp\left(-\alpha \,\mathrm{CE}_{\,t+1}\right) = E_t\left[\exp\left(-\alpha U_{t+1}\right)\right] \Longrightarrow \quad (74)$$

$$\mathrm{CE}_{\,t+1} = -\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha U_{t+1}\right)\right]\right), \quad (75)$$

  where U_{t+1} is the period-$t+1$ onward utility and $\mathrm{CE}_{\,t+1}$ is the certainty equivalent.

- The preferences over random consumption have the following recursive representation:

$$\begin{aligned} U_t &= U(c_t, \widetilde{c}_{t+1}, \widetilde{c}_{t+2}, \cdots) = W(c_t, U(\widetilde{c}_{t+1}, \widetilde{c}_{t+2}, \cdots)) \triangleq W(c_t, \mathrm{CE}_{\,t+1}) \\ &= \left\{ \begin{array}{c} (1-\beta)c_t^{1-1/\rho} \\ +\beta\left[-\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha U_{t+1}\right)\right]\right)\right]^{1-1/\rho} \end{array} \right\}^{\frac{1}{1-1/\rho}}. \end{aligned}$$

- The optimal solution to this problem can be found by solving the following Bellman equation

$$
\begin{aligned}
& v\left(a_t, y_t\right) && (76)\\
= & \max_{c_t}\left\{\begin{array}{c}(1-\beta) c_t^{1-1/\rho}+ \\ \beta\left[-\frac{1}{\alpha} \ln \left(E_t\left[\exp \left(-\alpha v\left(a_{t+1}, y_{t+1}\right)\right)\right]\right)\right]^{1-1/\rho}\end{array}\right\}^{1/(1-1/\rho)},
\end{aligned}
$$

subject to:

$$
\begin{aligned}
a_{t+1} &= R\left(a_t-c_t\right)+y_{t+1}, \\
y_{t+1} &= \mu y_t+(1-\mu) \bar{y}+\varepsilon_{t+1} && (77)
\end{aligned}
$$

given $a_0=y_0$, where ε_{t+1} is an iid innovation with mean 0 and variance ω^2.

- The no-Ponzi-game (nPg) condition is satisfied: $\lim_{j\to\infty}\left(\frac{a_{t+j}}{R^j}\right)\geq 0$.

- Guess the value function takes the following form

$$v(a_t, y_t) = Aa_t + By_t + C. \tag{78}$$

- Substituting it into (76), we have

$$\begin{aligned}
& -\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha v\left(a_{t+1}, y_{t+1}\right)\right)\right]\right) \\
= & -\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha\left(Aa_{t+1} + By_{t+1} + C\right)\right)\right]\right) \\
= & -\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha A\left[R\left(a_t - c_t\right) + y_{t+1}\right] - \alpha By_{t+1} - \alpha C\right)\right]\right) \\
= & -\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(\begin{array}{c} -\alpha AR\left(a_t - c_t\right) \\ -\alpha\left(A+B\right)\left[\mu y_t + \left(1-\mu\right)\overline{y} + \varepsilon_{t+1}\right] - \alpha C \end{array}\right)\right]\right. \\
= & AR\left(a_t - c_t\right) + \left(A+B\right)\left[\mu y_t + \left(1-\mu\right)\overline{y}\right] + C \\
& -\frac{1}{\alpha}\ln\left(E_t\left[\exp\left(-\alpha\left(A+B\right)\varepsilon_{t+1}\right)\right]\right) \\
= & AR\left(a_t - c_t\right) + \left(A+B\right)\left[\mu y_t + \left(1-\mu\right)\overline{y}\right] + C - \frac{1}{2}\alpha\left(A+B\right)^2\omega^2
\end{aligned}$$

- The FOC for c_t is

$$(1-\beta)c_t^{-1/\rho} = \beta AR \left[\begin{array}{c} AR\left(a_t - c_t\right) + \left(A+B\right)\left[\mu y_t + \left(1-\mu\right)\overline{y}\right] \\ +C - \frac{1}{2}\alpha\left(A+B\right)^2\omega^2 \end{array} \right]^{-1/\rho}$$

which means that

$$\begin{aligned} c_t &= \frac{\left(\frac{1-\beta}{\beta AR}\right)^{\rho} AR}{1+\left(\frac{1-\beta}{\beta AR}\right)^{\rho} AR} a_t \\ &\quad + \frac{\left(\frac{1-\beta}{\beta AR}\right)^{\rho}}{1+\left(\frac{1-\beta}{\beta AR}\right)^{\rho} AR} \left\{ \begin{array}{c} \left(A+B\right)\left[\mu y_t + \left(1-\mu\right)\overline{y}\right] \\ +C - \frac{1}{2}\alpha\left(A+B\right)^2\omega^2 \end{array} \right\} \\ &= H\left(ARa_t + \left\{ \begin{array}{c} \left(A+B\right)\left[\mu y_t + \left(1-\mu\right)\overline{y}\right] \\ +C - \frac{1}{2}\alpha\left(A+B\right)^2\omega^2 \end{array} \right\} \right) \qquad (79) \end{aligned}$$

where $H = \frac{\left(\frac{1-\beta}{\beta AR}\right)^{\rho}}{1+\left(\frac{1-\beta}{\beta AR}\right)^{\rho} AR}$.

- Substituting (79) into the Bellman equation gives

$$
\begin{aligned}
& Aa_t + By_t + C \\
= & \left\{ \begin{array}{c} (1-\beta) c_t^{1-1/\rho} \\ +\beta \left[\begin{array}{c} AR\left(a_t - c_t\right) + \\ (A+B)\left[\mu y_t + (1-\mu)\overline{y}\right] + \\ C - \frac{1}{2}\alpha\left(A+B\right)^2 \omega^2 \end{array} \right]^{1-1/\rho} \end{array} \right\}^{1/(1-1/\rho)} \\
= & \left[(1-\beta) c_t^{1-1/\rho} + \beta \left(-ARc_t + \frac{1}{H} c_t \right)^{1-1/\rho} \right]^{1/(1-1/\rho)} \\
= & \left[(1-\beta) + \beta \left(-AR + \frac{1}{H} \right)^{1-1/\rho} \right]^{1/(1-1/\rho)} c_t \\
= & \left[(1-\beta) \frac{1}{ARH} \right]^{1/(1-1/\rho)} H \left(ARa_t + \left\{ \begin{array}{c} (A+B)\left[\mu y_t + (1-\mu)\overline{y}\right] \\ +C - \frac{1}{2}\alpha\left(A+B\right)^2 \omega^2 \end{array} \right. \right. \\
= & \left(\frac{1-\beta}{AR} \right)^{1/(1-1/\rho)} H^{(1/\rho)/(1/\rho-1)} \left(\begin{array}{c} ARa_t + \\ \left\{ \begin{array}{c} (A+B)\left[\mu y_t + (1-\mu)\overline{y}\right] \\ +C - \frac{1}{2}\alpha\left(A+B\right)^2 \omega^2 \end{array} \right. \end{array} \right.
\end{aligned}
$$

- When $\rho \neq 1$, the undetermined coefficients, A, B, and C, must be such that the following equalities hold:

$$
\begin{aligned}
1 &= \left(\frac{1-\beta}{AR}\right)^{1/(1-1/\rho)} H^{(1/\rho)/(1/\rho-1)} R && (80) \\
B &= \left(\frac{1-\beta}{AR}\right)^{1/(1-1/\rho)} H^{(1/\rho)/(1/\rho-1)} (A+B)\mu && (81) \\
C &= \left(\frac{1-\beta}{AR}\right)^{1/(1-1/\rho)} H^{(1/\rho)/(1/\rho-1)} \left[\begin{array}{c} (A+B)(1-\mu)\bar{y}+C \\ -\frac{1}{2}\alpha(A+B)^2\omega^2 \end{array}\right] && (82)
\end{aligned}
$$

- Dividing (81) and (82) by (80), we have

$$
\begin{aligned}
\frac{A+B}{A} &= \frac{R}{R-\mu}, && (83) \\
\frac{C}{A} &= \frac{1}{R-1}\frac{R}{R-\mu}\left[(1-\mu)\bar{y}-\frac{1}{2}\alpha(A+B)\omega^2\right] && (84)
\end{aligned}
$$

- From (80), we can express

$$\begin{aligned} H^{-(1/\rho)/(1/\rho-1)} &= \left(\frac{1-\beta}{AR}\right)^{1/(1-1/\rho)} R \\ &= (1-\beta)^{1/(1-1/\rho)} R^{(1/\rho)(1/\rho-1)} A^{1/(1/\rho-1)} \\ &\Rightarrow H = (1-\beta)^{\rho} R^{-1} A^{-\rho}. \qquad (85) \end{aligned}$$

- Substituting it into $H = \frac{\left(\frac{1-\beta}{\beta AR}\right)^{\rho}}{1+\left(\frac{1-\beta}{\beta AR}\right)^{\rho} AR}$, we can solve for

$$1 + (1-\beta)^{\rho} A^{1-\rho} \left(\frac{1}{\beta R}\right)^{\rho} R = \left(\frac{1}{\beta R}\right)^{\rho} R \Rightarrow$$

$$A = \left[\frac{\left(\frac{1}{\beta R}\right)^{\rho} R - 1}{(1-\beta)^{\rho} \left(\frac{1}{\beta R}\right)^{\rho} R}\right]^{1/(1-\rho)} > 0, \qquad (86)$$

given that $\left(\frac{1}{\beta R}\right)^{\rho} R - 1 > 0$, i.e., $\beta^{\rho} R^{\rho-1} < 1$. From (83) and (84), we can easily determine B and C.

- Substituting (86) into (85), (83), and (84) gives

$$H = (1-\beta)^{\rho} R^{-1} \left[\frac{\left(\frac{1}{\beta R}\right)^{\rho} R - 1}{(1-\beta)^{\rho} \left(\frac{1}{\beta R}\right)^{\rho} R} \right]^{-\rho/(1-\rho)}$$

$$A + B = \frac{R}{R-\mu} \left[\frac{\left(\frac{1}{\beta R}\right)^{\rho} R - 1}{(1-\beta)^{\rho} \left(\frac{1}{\beta R}\right)^{\rho} R} \right]^{1/(1-\rho)}$$

$$C = \frac{R}{(R-1)(R-\mu)} \left[\begin{array}{c} (1-\mu)\,\overline{y} \\ -\frac{1}{2}\alpha\,(A+B)\,\omega^2 \end{array} \right] \left[\frac{\left(\frac{1}{\beta R}\right)^{\rho} R - 1}{(1-\beta)^{\rho} \left(\frac{1}{\beta R}\right)^{\rho} R}\right.$$

- Substituting these expressions into (79),

$$c_t = \left[1 - (\beta R)^{\rho} / R\right] \left\{ \begin{array}{c} a_t + \frac{\mu}{R-\mu} y_t \\ + \frac{R}{R-1} \frac{1}{R-\mu} \left[(1-\mu)\,\overline{y} - \frac{1}{2} \frac{\alpha A R}{R-\mu} \omega^2 \right] \end{array} \right\}. \tag{87}$$

- Combining (87) with the budget constraint gives the dynamics of consumption:

$$c_{t+1} = (\beta R)^{\rho} c_t + \left[1 - (\beta R)^{\rho} / R\right] \frac{R}{R-\mu} \varepsilon_{t+1} + const. \tag{88}$$

  We need to assume that $\beta R > 1$ to make sure that optimal consumption would grow at the gross rate $(\beta R)^{\rho}$ under certainty. This assumption can help eliminate negative consumption.
- (88) means that no contemporaneous variable beyond current consumption has predictive power for future consumption.

Economic Implications

- The MPC out of total wealth (non-human plus human wealth) is constant:

$$MPC = 1 - (\beta R)^{\rho} / R, \tag{89}$$

which does *not* depend on the degree of risk aversion (α) but depends on the degree of intertermporal substitution (ρ):

$$\frac{\partial (MPC)}{\partial \rho} = -\left[(\beta R)^{\rho} / R\right] \ln (\beta R) < 0 \tag{90}$$

because we assumed that $\beta R > 1$. Note that when $\rho = 1$, $MPC = 1 - \beta$.

- The precautionary saving premium depends on both parameters $(\rho$ and $\alpha)$:

$$PS = \left[1 - (\beta R)^{\rho} / R\right] \left\{ \frac{\alpha A R^2 \omega^2}{2 (R-1)(R-\mu)^2} \right\} \Rightarrow \tag{91}$$

$$\frac{\partial (PS)}{\partial \omega^2} = > 0, \frac{\partial (PS)}{\partial \alpha} > 0, \frac{\partial (PS)}{\partial \mu} > 0. \tag{92}$$

- We now assume that the utility function is CRRA: $\frac{c_{t+i}^{1-\gamma}-1}{1-\gamma}$. The optimizing problem of consumers is:

$$v\left(w_t\right)=\max_{\left\{c_{t+i},w_{t+1+i}\right\}} E_t\left[\sum_{i=0}^{\infty}\beta^i\frac{c_{t+i}^{1-\gamma}-1}{1-\gamma}\right] \tag{93}$$

subject to the constraint (for $i=0,\cdots,\infty$):

$$w_{t+1+i}=R\left(w_{t+i}-c_{t+i}+y_{t+i}\right), \tag{94}$$

w_t given, where $\beta=\frac{1}{1+\rho}$ and $R=1+r$.

- We cannot solve this problem explicitly to obtain the closed-form consumption function ($c\left(w,y\right)$) and consumption dynamics (Δc_{t+1}) as in the linear-quadratic and CARA utility cases.

Solution Method (Optional)

- It is difficult to solve the model with CRRA utility and risky labor income. Note that the difference of this model with the RBC model is that in this model we cannot define the traditional steady state around which we can linearize or log-linearize the model. Carroll (1997) solved this type of models using nonlinear computational methods.
- It is also possible to solve the model using the log-linearization method proposed by Campbell: Log-linearize both the nonlinear Euler equation and the budget constraint around the *endogenous* steady state, $E\left[c_t - w_t\right]$ and $E\left[y_t - w_t\right]$ that determine the log-linearization parameters, then determine the consumption function $c_t\left(w_t, y_t\right)$, and eventually determine themselves. That is, we need to write an algorithm to make the loop converge.

A Simplified Merton Model with One Risky Asset

- As above, we assume that the utility function is CRRA ($\gamma > 0$):

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

and the budget constraint is

$$a_{t+1} = R_{t+1}(a_t + y_t - c_t), \tag{95}$$

where $a_t + y_t - c_t$ is saving, and R_{t+1} is the iid return to the only risky asset and

$$\log R_{t+1} \sim N(r, \omega^2). \tag{96}$$

Without loss of generality, we focus on the case in which there is no labor income: $y_t = 0$. (It can be rationalized by assuming that labor income is fully diversified, i.e., it is marketable and can be reflected in a_t.) Note that the Euler equation holds:

$$1 = \beta E_t \left[R_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]. \tag{97}$$

Solving for the Consumption Function

- Suppose that the consumption function (i.e., optimal consumption) takes the following form:

$$c_t = \kappa a_t, \tag{98}$$

where κ is the undetermined coefficient and will be pinned down later.

- Substituting this guessed solution into the Euler equation gives

$$\begin{aligned}
1 &= \beta E_t \left[R_{t+1} \left(\frac{a_{t+1}}{a_t} \right)^{-\gamma} \right] \\
&= \beta E_t \left[R_{t+1} \left(\frac{R_{t+1} \left(a_t - c_t \right)}{a_t} \right)^{-\gamma} \right] &(99) \\
&= \beta E_t \left[R_{t+1} \left(\frac{R_{t+1} \left(1 - \kappa \right) a_t}{a_t} \right)^{-\gamma} \right] &(100) \\
&= \beta E_t \left[R_{t+1} \left(R_{t+1} \left(1 - \kappa \right) \right)^{-\gamma} \right] &(101) \\
&= \beta E_t \left[R_{t+1}^{1-\gamma} \left(1 - \kappa \right)^{-\gamma} \right], &(102)
\end{aligned}$$

- (Conti.) Using the last equation, we have

$$\kappa = 1 - \left[\beta E_t\left[R_{t+1}^{1-\gamma}\right]\right]^{1/\gamma}. \qquad (103)$$

- Given the specification of the return process (96), we can solve for κ. To proceed, we need to use the following fact:

Fact

If $\log X \sim N(\mu, \sigma^2)$, *then*

$$\begin{aligned}\log E[X] &= E[\log X] + \frac{1}{2}\sigma^2 = \mu + \frac{1}{2}\sigma^2 \text{ or} \\ E[X] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right)\end{aligned}$$

- Since $\log R_{t+1}^{1-\gamma} = (1-\gamma)\log R_{t+1}$,

$$\log E_t\left[R_{t+1}^{1-\gamma}\right] = (1-\gamma)\,r + \frac{1}{2}(1-\gamma)^2\,\omega^2 \Longrightarrow \qquad (104)$$

$$E_t\left[R_{t+1}^{1-\gamma}\right] = \exp\left((1-\gamma)\,r + \frac{1}{2}(1-\gamma)^2\,\omega^2\right). \qquad (105)$$

- (Conti.) Substituting it into (103) gives:

$$\kappa = 1-\beta^{1/\gamma}\exp\left(\left(\frac{1}{\gamma}-1\right)r+\frac{1}{2\gamma}\left(1-\gamma\right)^{2}\omega^{2}\right) \quad (106)$$

$$\simeq r-\left(1/\gamma\right)\left(r-\theta\right)-\frac{1}{2\gamma}\left(1-\gamma\right)^{2}\omega^{2}, \quad (107)$$

where $\beta=\frac{1}{1+\theta}$ and ω^2 is a small number (in the U.S. data, it is less than 0.15^2).
- Implication: given the level of a_t, as uncertainty goes up (larger ω^2), the level of consumption falls, reflecting the precautionary savings motive. κ here is just the marginal propensity to consume out of wealth (MPC).
- Note that we can also solve this problem using the Bellman equation:

$$v\left(a_t\right) = \max_{c_t}\left\{u\left(c_t\right)+\beta E_t\left[v\left(a_{t+1}\right)\right]\right\}, \quad (108)$$

$$s.t.\ a_{t+1} = R_{t+1}\left(a_t-c_t\right). \quad (109)$$

Hint: You can guess the value function that is similar to $u\left(c_t\right)$: $v\left(a_t\right)=A_0+A_1\frac{a_t^{1-\gamma}}{1-\gamma}$.

The Welfare Costs of Consumption Fluctuations

- We now examine what the effect on welfare would be if all consumption volatility could be eliminated. Following Lucas (2003), we consider a representative agent economy:

$$E\left[\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma}\right],$$

where we assume that the agent is endowed with the stochastic consumption stream

$$c_t = A\exp\left(\mu t\right)\exp\left(-\frac{1}{2}\sigma^2\right)\varepsilon_t \tag{110}$$

and ε_t follows a log-normal distribution

$$\log\left(\varepsilon_t\right)\sim N\left(0,\sigma^2\right), \tag{111}$$

which means that

$$E\left[\exp\left(-\frac{1}{2}\sigma^2\right)\varepsilon_t\right] = \exp\left(-\frac{1}{2}\sigma^2\right)E\left[\varepsilon_t\right] = 1. \tag{112}$$

- Therefore, mean consumption at time t is

$$E[c_t] = A\exp(\mu t). \quad (113)$$

- Since this agent is risk averse, he or she would obviously prefer a deterministic consumption path to a risky path with the same mean. We quantify this utility difference by multiplying the risky path by the constant factor $1+\lambda$ in all dates and states, choosing λ so that the agent is indifferent between the *deterministic* stream and the *compensated, risky* stream. That is, λ is chosen to solve

$$\begin{aligned} E\left[\sum_{t=0}^{\infty}\beta^t\frac{((1+\lambda)c_t)^{1-\gamma}}{1-\gamma}\right] &= \sum_{t=0}^{\infty}\beta^t\frac{(A\exp(\mu t))^{1-\gamma}}{1-\gamma}, \quad (114) \\ (1+\lambda)^{1-\gamma}\sum_{t=0}^{\infty}\beta^t E\left[c_t^{1-\gamma}\right] &= A^{1-\gamma}\sum_{t=0}^{\infty}\beta^t\exp(\mu(1-\gamma)t), \end{aligned}$$

where c_t is given in (110).

- Using the facts that

$$E\left[c_t^{1-\gamma}\right] = E\left[\begin{array}{c} A^{1-\gamma}\exp\left(\mu\left(1-\gamma\right)t - \frac{1}{2}\left(1-\gamma\right)\sigma^2\right)\cdot \\ \exp\left(\left(1-\gamma\right)\log\left(\varepsilon_t\right)\right) \end{array}\right]$$

$$E\left[\left(1-\gamma\right)\log\left(\varepsilon_t\right)\right] = 0$$

$$\mathrm{var}\left[\left(1-\gamma\right)\log\left(\varepsilon_t\right)\right] = \left(1-\gamma\right)^2\sigma^2,$$

we have

$$\begin{aligned} E\left[c_t^{1-\gamma}\right] &= A^{1-\gamma}\exp\left(\mu\left(1-\gamma\right)t - \frac{1}{2}\left(1-\gamma\right)\sigma^2 + \frac{1}{2}\left(1-\gamma\right)^2\sigma^2\right) \\ &= A^{1-\gamma}\exp\left(\mu\left(1-\gamma\right)t\right)\exp\left(-\frac{1}{2}\gamma\left(1-\gamma\right)\sigma^2\right). \end{aligned} \quad (115)$$

- Cancelling, taking logs, and collecting terms gives

$$
\begin{aligned}
& A^{1-\gamma}(1+\lambda)^{1-\gamma}\exp\left(-\frac{1}{2}\gamma(1-\gamma)\sigma^2\right)\sum_{t=0}^{\infty}\beta^t\exp(\mu(1-\gamma)t) \\
= \; & A^{1-\gamma}\sum_{t=0}^{\infty}\beta^t\exp(\mu(1-\gamma)t), \Longrightarrow \\
& (1+\lambda)^{1-\gamma}\exp\left(-\frac{1}{2}\gamma(1-\gamma)\sigma^2\right)=1\Longrightarrow \\
& \lambda\simeq\frac{1}{2}\gamma\sigma^2. \qquad (116)
\end{aligned}
$$

- Implications: The compensation parameter, λ, the welfare gain from eliminating consumption risk–depends on:
    - The amount of risk that is present, σ^2.
    - The degree of risk aversion γ.

Numerical Example

- We now view this agent as a representative of the U.S. consumers. To obtain λ, we need estimates of (1) the variance σ^2 of the log of consumption about its trend, and of (2) the coefficient of relative risk aversion γ.
- Using annual U.S. data for the period 1947-2001, the standard deviation of the log of real, per capita consumption about a linear trend (σ) is 0.032. Estimates of the parameter γ in macroeconomics and asset pricing range from 1 (log utility) to 4.
- Using log utility,

$$\lambda \simeq \frac{1}{2}\gamma\sigma^2 = \frac{1}{2}0.032^2 = 0.0005.$$

which is about one twentieth of one percent of consumption. If $\gamma = 2$, $\lambda = 0.001$. These estimates are trivally small!

Caballero's Permanent Income Model in General Equilibrium

- The individual optimizing problem can be written as:

$$V(a_0, y_0) = \max_{\{c_t, a_t\}_{t=0}^{\infty}} \left\{ E_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t u(c_t) \right] \right\}, \quad (117)$$

$$s.t. a_{t+1} = (1+r) a_t + y_t - c_t, \quad (118)$$

$$y_t = \phi_0 + \phi_1 y_{t-1} + w_t, \ \ |\phi_1| < 1, \quad (119)$$

where $u(c_t) = -\exp(-\alpha c)/\alpha$ is a CARA utility with $\alpha > 0$, $\rho > 0$ is the agent's subjective discount rate, r is a constant rate of interest, and $w_t \sim N(0, \sigma^2)$, $\phi_0 = (1-\phi_1)\overline{y}$, $\overline{y} = E[y_t]$, and y_0 and a_0 are given.

Optimal Consumption

- Following the same procedure adopted above, we obtain:

$$c_t^* = r\left\{a_t + h_t + \frac{1}{\alpha r^2}\left[\ln\left(\frac{1+\rho}{1+r}\right) - \ln E_t\left[\exp\left(-r\alpha\phi w_{t+1}\right)\right]\right]\right\}, \quad (120)$$

  where

$$h_t \equiv \frac{1}{1+r}E_t\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right] = \phi\left(y_t + \frac{\phi_0}{r}\right), \quad (121)$$

  is human wealth and $\phi = 1/\left(1+r-\phi_1\right)$.

- In (120), $\ln\left(\frac{1+\rho}{1+r}\right)/\left(r\alpha\right)$ measures the relative importance of impatience and the interest rate in determining current consumption, and

$$\ln\left(E_t\left[\exp\left(-r\alpha\phi w_{t+1}\right)\right]\right)/\left(r\alpha\right) = \frac{1}{2}\left(r\alpha\right)\omega_\zeta^2,$$

  measures the amount of precautionary savings determined by the interaction of risk aversion and income uncertainty.

Another Approach to Solve this Model

- We can reduce the multivariate model to a univariate model with iid innovations to permanent income. Letting permanent income, $s_t = a_t + h_t$, be defined as a new state variable, we can reformulate the PIH model as

$$v(s_t) = \max_{c_t} \left\{ u(c_t) + \frac{1}{1+\rho} E_t \left[v(s_{t+1}) \right] \right\},$$

s.t.

$$s_{t+1} = (1+r) s_t - c_t + \zeta_{t+1}, \tag{122}$$

where the time $(t+1)$ innovation to permanent income can be written as

$$\zeta_{t+1} \equiv \frac{1}{1+r} \sum_{j=t+1}^{\infty} \left(\frac{1}{1+r} \right)^{j-(t+1)} (E_{t+1} - E_t) [y_j] = \phi w_{t+1}, \tag{123}$$

where $v(s_t)$ is the consumer's value function.

A Derivation of the New Budget Constraint

- Given $s_t = a_t + h_t = a_t + \phi\left(y_t + \frac{\phi_0}{r}\right)$, the original budget constraint can be rewritten as

$$
\begin{aligned}
& a_{t+1} + \phi y_{t+1} + \frac{\phi\phi_0}{r} \\
& = (1+r)\, a_t + y_t - c_t + \phi\left(\phi_0 + \phi_1 y_t + w_{t+1}\right) + \frac{\phi\phi_0}{r} \\
& = (1+r)\left(a_t + \phi y_t + \frac{\phi\phi_0}{r}\right) - c_t + \zeta_{t+1},
\end{aligned}
$$

which is just (122).

- Using the consumption function and the definition of the saving function,

$$
\begin{aligned}
d_t^* &= ra_t + y_t - c_t^* \\
&= ra_t + y_t - r\left[a_t + \phi y_t + \frac{\phi\phi_0}{r} + \frac{1}{r^2\alpha}\begin{pmatrix} \ln\left(\frac{1+\rho}{1+r}\right) \\ -\ln\left(E_t\left[\exp\left(-r\alpha\phi w_{t+1}\right)\right]\right)\end{pmatrix}\right] \\
&= \left(1-\phi_1\right)\phi\left(y_t - \overline{y}\right) + \frac{1}{r\alpha}\left[\ln\left(E_t\left[\exp\left(-r\alpha\zeta_{t+1}\right)\right]\right) - \ln\left(\frac{1+\rho}{1+r}\right)\right] \\
&= \left(1-\phi_1\right)\phi\left(y_t - \overline{y}\right) + \frac{1}{r\alpha}\left[\frac{1}{2}\left(r\alpha\right)^2\omega_\zeta^2 - \ln\left(\frac{1+\rho}{1+r}\right)\right], \qquad (124)
\end{aligned}
$$

where the first term captures the consumer's demand for savings "for a rainy day", the second term, $\frac{1}{2}r\alpha\omega_\zeta^2$, is the precautionary savings premium, and the third term $\frac{1}{r\alpha}\ln\left(\frac{1+\rho}{1+r}\right)$ captures the dissaving effects of impatience.

Existence of General Equilibrium

- Wang (2003) assumes that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, of total mass normalized to one, with each agent solving the optimal consumption and savings problem specified above. Note that the risk-free asset in our model is a pure-consumption loan and is in zero net supply. The initial cross-sectional distribution of permanent income is assumed to be the stationary distribution $\Phi(\cdot)$. By the law of large numbers, aggregate permanent income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ are constant over time.

Proposition

The total savings demand "for a rainy day" in the precautionary savings model with RI equals zero for any positive interest rate. That is, $F_t(r) = \int_{y_t} f_t(r)\, d\Phi(y_t) = 0$, *for* $r > 0$.

- (Conti.) Therefore, for $r > 0$, the expression for total savings at time t is

$$D(\theta, r) \equiv \frac{1}{2}(r\alpha)\,\omega_{\zeta}^{2} - \frac{1}{r\alpha}\ln\left(\frac{1+\rho}{1+r}\right). \tag{125}$$

The equilibrium in this economy is defined by an interest rate r^* satisfying:

$$D(\theta, r^*) = 0. \tag{126}$$

Proposition

There exists at least one equilibrium with an interest rate $r^* \in (0, \rho)$ in this precautionary-savings model. In any such equilibrium, each agent's consumption is described by the PIH, in that

$$c_t^* = r^* s_t. \tag{127}$$

Proof.

If $r > \rho$, both $\frac{1}{2}(r\alpha)\,\omega_\zeta^2$ and $-\frac{1}{r\alpha}\ln\left(\frac{1+\rho}{1+r}\right)$ are positive, which contradicts the equilibrium condition, $D(\theta, r^*) = 0$. Since $\frac{1}{2}(r\alpha)\,\omega_\zeta^2 - \frac{1}{r\alpha}\ln\left(\frac{1+\rho}{1+r}\right) < 0$ (> 0) when $r = 0$ $(r = \rho)$, the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \rho)$ such that $D(\theta, r^*) = 0$. From (120), we can obtain the individual's optimal consumption rule in general equilibrium as $c_t^* = r^* \widehat{s}_t$. □

Implication

- With an individual's constant total precautionary savings demand $\frac{1}{2}(r\alpha)\,\omega_{\zeta}^{2}$, for any $r > 0$, the equilibrium interest rate r^{*} must be such that each individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand:

$$\frac{1}{2}(r\alpha)\,\omega_{\zeta}^{2} = \frac{1}{r\alpha}\ln\left(\frac{1+\rho}{1+r}\right).$$