Lecture 8: Portfolio Choice and Consumption-based Asset Pricing

Yulei Luo

Econ HKU

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The Portfolio Choice Problem

- In the permanent income model we have discussed, we assumed that individuals can only invest in one risk free asset. However, individuals can invest in many assets including the risk free asset and risky assets having uncertain returns.
- In the presence of multiple assets, individuals face two optimal decisions: *consumption-saving decision* and *portfolio choice*.
- We start from a simple *static* portfolio choice problem. Consider an investor with wealth level w_0, who need to decide how to allocate his or her wealth in two assets: one is a risky asset with uncertain rate $\widetilde{r}$ and the other is a risk free asset that pays a certain return r_f.
- The investor's wealth at the end of the period is

$$w_1 = a\left(1+\widetilde{r}\right) + \left(w_0 - a\right)\left(1+r_f\right) = w_0\left(1+r_f\right) + a\left(\widetilde{r} - r_f\right),$$

where a is the amount of wealth invested in the risky asset.

- (Conti.) Thus the portfolio choice problem can be written as

$$\max_a E[u(w_1)] = \max_a E[u(w_0(1 + r_f) + a(\widetilde{r} - r_f))] \tag{1}$$

where E is the expectation operator, $u(\cdot)$ is the utility function defined on final wealth (Note that in the static setup, it is equivalent with consumption), and then $E[u(\cdot)]$ the expected utility.

- Under risk aversion ($u'' < 0$), the optimal solution for the above portfolio choice problem must satisfy:

$$E[u'(w_0(1 + r_f) + a(\widetilde{r} - r_f)) \cdot (\widetilde{r} - r_f)] = 0, \tag{2}$$

which describes the relationship between risk aversion and optimal asset allocation.

- Note that if $u' > 0, u'' < 0$, and a^* is the solution to (2), then

$$a^* \gtreqless 0 \Longleftrightarrow E(\widetilde{r}) \gtreqless r_f$$

Proof.

Define $W(a) = E[u(w_0(1 + r_f) + a(\tilde{r} - r_f))]$. (2) means that

$$
\begin{aligned}
W'(a) &= E[u'(w_0(1 + r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)] = 0, \\
W''(a) &= E[u''(w_0(1 + r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)^2] < 0
\end{aligned}
$$

because $u''(\cdot) < 0$. It follows that $a^* > 0$ if and only if

$$
W'(0) = E[u'(w_0(1 + r_f)) \cdot (\tilde{r} - r_f)] = u'(w_0(1 + r_f))E(\tilde{r} - r_f) > 0
$$

because a will have to increase from 0 to reach the equality in (2). □

- This result means that a risk averse investor will invest in the risky asset only if the expected return on the risky asset is greater than the risk free rate.

Optimal Asset Allocation: CRRA Case

- Start with $u(x) = \ln(x)$ and for simplicity assume that the return on the risky asset has two states ("down" or "up" in the stock market)

$$\widetilde{r} = \begin{cases} r_1, \text{ with probability } 1-\pi, \\ r_2, \text{ with probability } \pi \end{cases}.$$

and $r_2 > r_f > r_1$. Hence, the FOC can be rewritten as

$$E\left[\frac{\widetilde{r} - r_f}{w_0(1+r_f) + a(\widetilde{r} - r_f)}\right] = 0.$$

Using the distribution of the returns, we may eliminate the expectation operator:

$$\frac{\pi(r_2 - r_f)}{w_0(1+r_f) + a(r_2 - r_f)} + \frac{(1-\pi)(r_1 - r_f)}{w_0(1+r_f) + a(r_1 - r_f)} = 0, \tag{3}$$

which can be solved directly

$$\frac{a}{w_0} = \frac{(1+r_f)E\left[\widetilde{r} - r_f\right]}{(r_f - r_1)(r_2 - r_f)}, \tag{4}$$

Implications

- The fraction of wealth invested in the risky asset, $\frac{a}{w_0}$, *increases* with the risky premium, $E\left[\widetilde{r} - r_f\right]$, and *decreases* with the dispersion of the return of the risky asset, $(r_f - r_1)(r_2 - r_f)$, (Note that in this two-state case, this dispersion measures the volatility of asset returns).
- It is *independent* of the wealth level (w_0) in the CRRA case.
- A numerical example: suppose that $r_2 = 0.4$, $r_f = 0.05$, $r_1 = -0.2$, and $\pi = 0.5$, substituting these values into the optimal ratio above gives

$$\frac{a}{w_0} = 0.6.$$

The Effects of Risk Aversion on Optimal Asset Allocation

- Consider $u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$, $\gamma > 1$. Following the same procedure above, we have

$$\frac{\pi(r_2 - r_f)}{[w_0(1+r_f)+a(r_2-r_f)]^\gamma} + \frac{(1-\pi)(r_1-r_f)}{[w_0(1+r_f)+a(r_1-r_f)]^\gamma} = 0, \tag{5}$$

which implies that

$$\frac{a}{w_0} = \frac{(1+r_f)\left\{[(1-\pi)(r_f - r_1)]^{1/\gamma} - [\pi(r_2 - r_f)]^{1/\gamma}\right\}}{(r_1-r_f)[\pi(r_2-r_f)]^{1/\gamma} - (r_2-r_f)[(1-\pi)(r_f-r_1)]^{1/\gamma}} \tag{6}$$

- A numerical example: suppose that $\gamma = 3$,

$$\frac{a}{w_0} = 0.24 < 0.6, \tag{7}$$

which means that risk aversion reduces the optimal allocation in the risky asset.

Optimal Asset Allocation: CARA

- Consider $u(x) = -\exp(-\alpha x)$ (that is, $R_A(x) = \alpha$), the maximization problem becomes

$$\max_a E[-\exp(-\alpha[w_0(1+r_f) + a(\widetilde{r} - r_f)])]. \tag{8}$$

- The FOC is thus

$$E[\alpha(\widetilde{r} - r_f)\exp(-\alpha[w_0(1+r_f) + a(\widetilde{r} - r_f)])] = 0. \tag{9}$$

For simplicity, we also consider two-state distribution, the optimal *amount* invested in the risky asset is

$$a = \frac{1}{\alpha}\frac{1}{r_1 - r_2}\log\left(\frac{1-\pi}{\pi}\frac{r_f - r_1}{r_2 - r_f}\right), \tag{10}$$

which is *independent* of the wealth level w_0, i.e.,

$$\frac{da}{dw_0} = 0. \tag{11}$$

Note that $a > 0 \iff \frac{1-\pi}{\pi}\frac{r_f - r_1}{r_2 - r_f} \in (0, 1) \Longleftarrow \pi > 1/2$.

- Note that we may also examine the property of a (i.e., $\frac{da}{dw_0}$) from the FOC directly instead of solving for the explicit solution: Differentiating the FOC w.r.t. w_0 gives

$$E\left[\begin{array}{c}\alpha^2\left(\widetilde{r}-r_f\right)\exp\left(-\alpha\left[w_0(1+r_f)+a(\widetilde{r}-r_f)\right]\right)\cdot \\ \left[(1+r_f)+\frac{da}{dw_0}(\widetilde{r}-r_f)\right]\end{array}\right]=0, \quad (12)$$

which implies that

$$(1+r_f)\underbrace{E[\alpha^2(\widetilde{r}-r_f)\exp(-\alpha(w_0(1+r_f)+a(\widetilde{r}-r_f)))]}_{=0 \text{ (Implied by the FOC (9))}}$$
$$+E[\underbrace{\alpha^2(\widetilde{r}-r_f)^2}_{>0}\underbrace{\exp\left(-\alpha\left[w_0(1+r_f)+a\left(\widetilde{r}-r_f\right)\right]\right)}_{>0}\frac{da}{dw_0}] = 0, \quad (13)$$

which also means that $\frac{da}{dw_0}=0$.

Optimal Asset Allocation: Quadratic utility

- Consider $u(x) = x - \frac{1}{2}x^2$ (note that we impose that $x < 1$), the maximization problem becomes

$$\max_a E\left[w_0(1+r_f) + a(\widetilde{r} - r_f) - \frac{1}{2}\left[w_0(1+r_f) + a(\widetilde{r} - r_f)\right]^2\right]. \tag{14}$$

- The FOC is

$$E\left[(\widetilde{r} - r_f)\left\{1 - \left[w_0(1+r_f) + a(\widetilde{r} - r_f)\right]\right\}\right] = 0. \tag{15}$$

Differentiating it w.r.t. w_0 gives

$$\begin{aligned}
E\left[(\widetilde{r} - r_f)\left[-(1+r_f) - \frac{da}{dw_0}(\widetilde{r} - r_f)\right]\right] &= 0 \Longrightarrow \\
-(1+r_f)\underbrace{E[\widetilde{r} - r_f]}_{>0} - E\left[(\widetilde{r} - r_f)^2\frac{da}{dw_0}\right] &= 0 \Longrightarrow
\end{aligned}$$

$$\frac{da}{dw_0} = -\frac{(1+r_f)E[\widetilde{r} - r_f]}{E[(\widetilde{r} - r_f)^2]} < 0. \tag{16}$$

- Given the two-state distribution, the optimal a is

$$
\begin{aligned}
\pi(r_2 - r_f)\left(1 - \left[w_0(1 + r_f) + a(r_2 - r_f)\right]\right) & \\
+(1 - \pi)(r_1 - r_f)\left(1 - \left[w_0(1 + r_f) + a(r_1 - r_f)\right]\right) &= 0 \quad (17)
\end{aligned}
$$

which implies that

$$
a = \frac{\left[\pi(r_2 - r_f) - (1 - \pi)(r_1 - r_f)\right]\left[1 - w_0(1 + r_f)\right]}{\pi(r_2 - r_f)^2 - (1 - \pi)(r_1 - r_f)^2}. \quad (18)
$$

A More General Static Case

- We abstract from consumption and saving decisions here. The optimizing portfolio choice problem is:

$$\max_{\alpha_t} \frac{E_t\left[A_{t+1}^{1-\gamma}\right]}{1-\gamma}, \text{ subject to: } A_{t+1} = R_{t+1}^p A_t, \tag{19}$$

where the market portfolio includes two assets: one risky asset and one risk free asset, and

$$R_{t+1}^p = \alpha_t R_{t+1}^e + (1-\alpha_t) R^f,$$

which means

$$E_t\left[R_{t+1}^p\right] = R^f + \alpha_t \left(R_{t+1}^e - R^f\right) \text{ and } \operatorname{var}\left[R_{t+1}^p\right] = \alpha_t^2 \operatorname{var}\left[R_{t+1}^e\right] \tag{20}$$

- Using the fact for the lognormal variable:

$$\log E_t[X_{t+1}] = E_t[\log X_{t+1}] + \frac{1}{2}\operatorname{var}_t[\log X_{t+1}], \tag{21}$$

the above optimizing problem is equivalent to

$$\max_{\alpha_t} \log E_t\left[A_{t+1}^{1-\gamma}\right] = \max_{\alpha_t} (1-\gamma) E_t[a_{t+1}] + \frac{1}{2}(1-\gamma)^2 \operatorname{var}_t[a_{t+1}], \tag{22}$$

where $a_{t+1} = \log A_{t+1}$.

- Taking log on both sides of the constraint gives

$$a_{t+1} = r_{t+1}^p + a_t, \tag{23}$$

where $r_{t+1}^p = \log R_{t+1}^p$. Rewriting the optimizing problem:

$$\max_{\alpha_t} E_t\left[r_{t+1}^p\right] + \frac{1}{2}(1-\gamma)\operatorname{var}_t\left[r_{t+1}^p\right] \tag{24}$$

- Next, we need to approximate the return to the market portfolio:

$$r_{t+1}^{p} - r^{f} = \alpha_t (r_{t+1}^{e} - r^{f}) + \frac{1}{2}\alpha_t (1-\alpha_t)\omega^2. \tag{25}$$

because:

$$\begin{aligned}
R_{t+1}^{p} &= \alpha_t R_{t+1}^{e} + (1-\alpha_t) R^{f} \Longrightarrow \\
\frac{R_{t+1}^{p}}{R^{f}} &= 1 + \alpha_t \left(\frac{R_{t+1}^{e}}{R^{f}} - 1 \right) \Longrightarrow \\
r_{t+1}^{p} - r^{f} &= \log \left[1 + \alpha_t \left(\exp\left(r_{t+1}^{e} - r^{f} \right) - 1 \right) \right],
\end{aligned}$$

which gives a nonlinear relation between $r_{t+1}^{p} - r^{f}$ and $r_{t+1}^{e} - r^{f}$.

- This relation can be approximated using a second-order Taylor expansion around the point $r^e_{t+1} - r^f = 0$. The function $f\left(r^e_{t+1} - r^f\right) = \log\left[1 + \alpha_t\left(\exp\left(r^e_{t+1} - r^f\right) - 1\right)\right]$ is approximated as:

$$f\left(r^e_{t+1} - r^f\right) \simeq f(0) + f'(0)\left(r^e_{t+1} - r^f\right) + \frac{1}{2}f''(0)\left(r^e_{t+1} - r^f\right)^2, \tag{26}$$

  where $f'(0) = \alpha_t$ and $f''(0) = \alpha_t(1 - \alpha_t)$. In addition, replace $\left(r^e_{t+1} - r^f\right)^2$ by its expectation ω^2.
- Substituting this approximation back into the optimizing problem gives

$$\max_{\alpha_t} E_t\left[r^f + \alpha_t(r^e_{t+1} - r^f) + \frac{1}{2}\alpha_t(1 - \alpha_t)\omega^2\right] + \frac{1}{2}(1 - \gamma)\alpha_t^2\omega^2, \tag{27}$$

  The FOC is

$$\alpha_t = \frac{E\left[r^e_{t+1} - r^f\right] + \omega^2/2}{\gamma\omega^2}, \tag{28}$$

  which has the same implications as the CRRA case with the two-state distribution we discussed above.

The Joint Saving-Portfolio Choice Problem

- We have so far distinguished the consumption-saving decision and the portfolio allocation decisions. The two decisions, however, should be considered jointly. We now formalize the consumption-savings and portfolio choice problem:

$$\max_{\{a,s\}} E\left[u(w_0 - s) + \beta u(s(1+r_f) + a(\widetilde{r} - r_f))\right] \tag{29}$$

$$s.t. w_0 \geq s \geq 0, \tag{30}$$

where s denotes the total amount of saving and a is the amount invested in the risky asset.

- When the utility function is CRRA, the FOCs are:

$$-(w_0 - s)^{-\gamma} + \beta E\left[(s(1+r_f) + a(\widetilde{r} - r_f))^{-\gamma}(1+r_f)\right] = 0 \tag{31}$$

$$E\left[(s(1+r_f) + a(\widetilde{r} - r_f))^{-\gamma}(\widetilde{r} - r_f)\right] = 0 \tag{32}$$

Implications

- The first FOC compares the marginal utility today with the expected marginal utility tomorrow.
- For the second FOC,

$$E\left[\left(s(1+r_f)+a(\widetilde{r}-r_f)\right)^{-\gamma}(\widetilde{r}-r_f)\right]=0 \Longrightarrow \quad (33)$$

$$E\left[\left(s(1+r_f)+a(\widetilde{r}-r_f)\right)^{-\gamma}\widetilde{r}\right]=E\left[\left(s(1+r_f)+a(\widetilde{r}-r_f)\right)^{-\gamma}r_f\right], \quad (34)$$

which means that if investors are behaving optimally, a marginal investment at t in *any* asset should yield the same *expected* marginal increase in utility at $t+1$.

A More Realistic Infinite-horizon Consumption-Saving and Portfolio Choice Model

- The infinite-horizon optimizing problem:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^j \frac{c_t^{1-\gamma} - 1}{1-\gamma} \right] \tag{35}$$

subject to

$$a_{t+1} = R_{t+1}^p (a_t + y_t - c_t), \tag{36}$$

where $R_{t+1}^p = \alpha R_{t+1}^e + (1-\alpha) R^f$. It is difficult to solve this problem.

- You may refer to Campbell and Viceira (*Strategic Asset Allocation* 2002, Cambridge U. Press) for the details about how to solve this problem and some interesting discussions on long-term asset allocation, labor income risk, and optimal consumption and savings.

- Even if we cannot easily solve the full-fledged optimal consumption and portfolio choice model, we can still gain many interesting insights about the joint dynamics of the asset return and consumption dynamics by inspecting the Euler equation.
- We now assume that there are n risky assets such that

$$R_{t+1}^{p}=\sum_{j=1}^{n}\alpha^{j}R_{t+1}^{j}+\left(1-\sum_{j=1}^{n}\alpha^{j}\right)R^{f}. \tag{37}$$

- In this case, the Euler equations for all assets are

$$u'\left(c_{t}\right)=\frac{1}{1+\rho}E_{t}\left[R_{t+1}^{j}u'\left(c_{t+1}\right)\right],$$

where $j=f,1,\cdots,n$.

- Note that the Euler equations can be rewritten as

$$\begin{aligned} 1 &= \frac{1}{1+\rho} E_t \left[R^j_{t+1} \frac{u'(c_{t+1})}{u'(c_t)} \right] \\ &\equiv E_t \left[R^j_{t+1} M_{t+1} \right], \end{aligned} \tag{38}$$

where M_{t+1} is the stochastic discount factor applied at t to consumption in the following period. It is the intertemporal marginal rate of substitution, i.e., the discounted ratio of marginal utilities of consumption in any two subsequent periods.

- Using (38), we have the key result of consumption-based asset pricing:

$$\begin{aligned} E_t \left[R^j_{t+1} M_{t+1} \right] &= E_t \left[R^j_{t+1} \right] E_t \left[M_{t+1} \right] + \mathrm{cov}_t \left(R^j_{t+1}, M_{t+1} \right) \Longrightarrow \\ E_t \left[R^j_{t+1} \right] &= \frac{1}{E_t \left[M_{t+1} \right]} \left[1 - \mathrm{cov}_t \left(R^j_{t+1}, M_{t+1} \right) \right]. \end{aligned} \tag{39}$$

- (conti.) In the case of the risk free asset, we have

$$R^f = \frac{1}{E_t\left[M_{t+1}\right]}. \tag{40}$$

- Combining (39) with (40), we have

$$E_t\left[R^j_{t+1}\right] - R^f = -R^f \operatorname{cov}_t\left(R^j_{t+1}, M_{t+1}\right), \tag{41}$$

  which means that: In equilibrium, the risky asset j whose return has a *negative* correlation with the SDF yields an expected return *higher* than R^f.
- This asset is risky for the investor because it yields lower returns when the marginal utility of consumption relatively high due to a relatively low level of consumption. In equilibrium investors are still willing to hold this asset only if such risk can be compensated by a premium determined by an expected return higher than the risk free rate R^f.

Implications of CRRA Utility

- (Conti.) Assume the CRRA utility $\frac{c_t^{1-\gamma}-1}{1-\gamma}$, we have

$$E_t\left[R_{t+1}^j\right] - R^f = -R^f \operatorname{cov}_t\left(R_{t+1}^j, \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right). \tag{42}$$

- Using the facts that (1) If $\frac{\Delta c_{t+1}}{c_t}$ is small,

$$\frac{c_{t+1}}{c_t} \simeq 1 + \Delta \log c_{t+1}. \tag{43}$$

(2) If x is small,

$$(1+x)^n \simeq 1 + nx, \tag{44}$$

(42) can be written as

$$\begin{aligned} E_t\left[R_{t+1}^j\right] - R^f &\simeq -\beta R^f \operatorname{cov}_t\left(R_{t+1}^j, 1-\gamma\Delta\log c_{t+1}\right) \\ &\simeq \operatorname{cov}_t\left(R_{t+1}^j, \gamma\Delta\log c_{t+1}\right) \end{aligned} \tag{45}$$

Note that βR^f is close to 1 if $\frac{\Delta c_{t+1}}{c_t}$ is small.

First Look: The Equity Premium Puzzle

- Taking unconditional expectations on both sides of (45) gives

$$
\begin{aligned}
E\left[R^j_{t+1}\right] - R^f &\simeq \operatorname{cov}\left(R^j_{t+1}, \gamma \Delta \log c_{t+1}\right) \qquad (46) \\
&= \gamma \operatorname{corr}\left(R^j_{t+1}, \Delta \log c_{t+1}\right) \operatorname{sd}\left(R^j_{t+1}\right) \operatorname{sd}\left(\Delta \log c_{t+1}\right)
\end{aligned}
$$

- Mehra and Prescott (1985) show that it is difficult to reconcile observed returns on stocks and bonds with equation (45). As documented in Campbell (2003), in the U.S. data from $1947.2 - 1998.4$:

$$
\begin{aligned}
\operatorname{corr}\left(R^j_{t+1}, \Delta \log c_{t+1}\right) &= 0.34, \qquad (47) \\
\operatorname{sd}\left(R^j_{t+1}\right) &= 15.6\%, \qquad (48) \\
\operatorname{sd}\left(\Delta \log c_{t+1}\right) &= 1.1\%, \qquad (49) \\
E\left[R^j_{t+1}\right] - R^f &= 7\%, \qquad (50)
\end{aligned}
$$

which means that $\gamma = 120$! It is highly unrealistic.

- To gain some idea about what plausible values of γ are, consider the following gamble: You must choose between a gamble in which you consume \$50000 in the rest of your life with probability 0.5 and \$100000 with probability 0.5, or consuming some amount X with certainty. The CRRA, γ, determines the value of X which would make you indifferent between consuming X or being exposed to the risky gamble.
- E.g., if $\gamma = 0$, then you have no risk aversion at all and will will be indifferent between \$75000 with certainty and the 50/50 gamble with expected value of \$75000. Here are the values of X associated with different γ:

γ	X
1	70711
5	58565
10	53991
30	51209
∞	50000

Consumption-based Capital Asset Pricing Model (C-CAPM)

- In reality, assets are traded every period, as new information becomes available, and decisions are made sequentially. Consequently, today's decisions affect tomorrow's opportunities. CCAPM can be used to capture these dynamic features and to price assets in such an environment.
- Another advantage: This theory provides a way to link the real economy (aggregate output and consumption) and financial markets (asset prices and returns).
- Lucas (1978) first developed this CCAPM theory. It is an endowment economy (i.e., no production decisions) and allows recursive security trading.

Lucas (1978)'s Asset Pricing Model

- Imagine the security as representing ownership of a fruit tree where the nonstorable output varies over time.
- Suppose that there is an economy with N *identical* agents with RE. That is, each agent's expectations are conditional on all available information (including the structure of the economy, the output process, etc.)
- Suppose that all output is obtained from an asset which produces a stochatic endowment of perishable consumption goods for each unit of the asset the agent owns at the beginning of t.
- If an agent owns z_t units of the asset at the beginning of t, he receives an endowment of $z_t y_t$ of the consumption good and y_t is identical for each unit of the asset held by an agent and is an *exogenous* stochastic process.
- Assume that they have identical endowments. Since the agents have identical preferences, they will make the same decisions given the state of the economy.

- (Conti.) The typical agent chooses optimal holdings of securities and consumes dividends:

$$\max_{\{c_t, z_{t+1}\}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \tag{51}$$

$$s.t.\ c_t + p_t z_{t+1} \leq z_t y_t + p_t z_t, \qquad \forall t, \tag{52}$$

  where p_t is the period t real price of the security in terms of consumption and z_t is the agent's beginning-of-period t holdings of security. The expectation operator applies across all possible states of y.
- In the this economy:
  - Financial markets are in equilibrium iff at the prevailing price, supply equals demand and the equilibrium price is that price at which the agents wishes to hold exactly the amount of the securities present in the economy.
  - Assume that the total supply of assets is N. With N agents wantin the same number of assets, we must have $z_t = 1$ for all t and for every agent *in equilibrium*.
  - The total net supply of I-owe-You (IOU) type of contract (inside bonds) must be zero; otherwise, there is no equilibrium.

- Output is an *exogenously* stochastic process. We may assume that the output process follows a n-state probability transition. E.g., y_t could take three states: $\begin{bmatrix} y^1 & y^2 & y^3 \end{bmatrix}$ and it follows a probability transition matrix to switch from one state to another state:

Table: Three-state probability transition matrix

output in $t+1$

output in t $\quad \Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$

where $\pi_{ij} = prob\left(y_{t+1} = y^j | y_t = y^i\right)$ for any t.

- Note this discrete-state distribution can be approximated from an $\text{AR}(1)$ process.

Approximating an AR(1) Process

- Given the following AR(1) process with *continuous* states,

$$y_{t+1} = \rho y_t + (1-\rho)\,\overline{y} + \varepsilon_{t+1}, \tag{53}$$

we can numerically approximate it with finite *discrete* states. The Matlab codes posted on the course website can used to approximate the AR process: `discretizationAR1.m` and `Tauchen.m`.

- E.g., given that $\rho = 0.9$, $\omega = 0.1$, $\overline{y} = 1$, and $n = 3$, running the discretization code yields:

$$\begin{bmatrix} y^1 & y^2 & y^3 \end{bmatrix} = \begin{bmatrix} 0.5412 & 1 & 1.4588 \end{bmatrix}, \tag{54}$$

$$\Pi = \begin{bmatrix} 0.9668 & 0.0332 & 0.0000 \\ 0.0109 & 0.9782 & 0.0109 \\ 0.0000 & 0.0332 & 0.9668 \end{bmatrix}. \tag{55}$$

Using the Bellman equation to Solve the Model

- Define the value function as

$$v(m_s) = \max E_s\left[\sum_{t=s}^{\infty}\beta^{t-s}u(c_t)\right] \quad (56)$$

$$s.t. \quad c_t + p_t z_{t+1} \leq z_t y_t + p_t z_t \quad (57)$$

where m_s is the beginning-of-period s wealth:

$$m_s = (y_s + p_s)\, z_s, \quad (58)$$

and rewrite the budget constraint as

$$c_s + p_s z_{s+1} = m_s \quad (59)$$

- The Bellman equation can be written as

$$
\begin{aligned}
v(m_s) &= \max_{\{c_s\}} \{u(c_s) + \beta E_s [v(m_{s+1})]\} && (60) \\
&= \max_{\{c_s\}} \{u(c_s) + \beta E_s [v((y_{s+1} + p_{s+1}) z_{s+1})]\} && (61) \\
&= \max_{\{c_s\}} \left\{ u(c_s) + \beta E_s \left[v((y_{s+1} + p_{s+1}) \frac{m_s - c_s}{p_s}) \right] \right\} && (62)
\end{aligned}
$$

- The FOC is:

$$
u_1(c_s) - \beta E_s \left[V_1(m_{s+1}) \frac{y_{s+1} + p_{s+1}}{p_s} \right] = 0 \quad (63)
$$

- The envelop theorem is:

$$
v_1(m_s) = \beta E_s \left[v_1(m_{s+1}) \frac{y_{s+1} + p_{s+1}}{p_s} \right], \quad (64)
$$

which means that $u_1(c_s) = v_1(m_s)$ and $u_1(c_{s+1}) = v_1(m_{s+1})$.

- Substituting this Envelop condition into the FOC gives the Euler equation:

$$u_1(c_s)p_s = \beta E_s\left[u_1(c_{s+1})\left(y_{s+1} + p_{s+1}\right)\right]. \tag{65}$$

- Given the functional dependence on the output state variables, the Euler equation can be written as

$$u_1\left(c_s\left(y^i\right)\right)p_s\left(y^i\right) = \beta\sum_j \pi_{ij}\left[u_1\left(c_{s+1}\left(y^j\right)\right)\left(y^j + p_{s+1}(y^j)\right)\right], \forall i \tag{66}$$

- Economic implications:
  - The LHS: the utility loss in period t associated with the purchase of an additional unit of the security.
  - The RHS: the *expected discounted* utility gain associated with selling the extra unit of the security.
  - If this equality is not satisfied, the agent will try either to increase or to decrease his holdings of the security to increase his utility.

In Equilibrium

- $z_t = z_{t+1} = \cdots = 1$, in other words, every agents owns the same number of trees, 1.
- $c_t = y_t$, that is, ownership of the entire security entitles the agent to all the economy's output.
- $u_1(c_s)p_s = \beta E_s \left[u_1(c_{s+1}) \left(y_{s+1} + p_{s+1} \right) \right]$ are optimal given the prevailing prices. Substituting $c_t = y_t$ into this Euler equation gives:

$$u_1(y_s)p_s = \beta E_s \left[u_1(y_{s+1}) \left(y_{s+1} + p_{s+1} \right) \right], \tag{67}$$

which is the fundamental equation of the consumption-based CAPM.

- A recursive substitution of (67) into itself yields

$$p_s = E_s \sum_{j=1}^{\infty} \left[\beta^j \frac{u_1\left(y_{s+j}\right)}{u_1\left(y_s\right)} y_{s+j} \right], \tag{68}$$

  where $M_{s,s+j} = \beta^j \frac{u_1\left(y_{s+j}\right)}{u_1\left(y_s\right)}$ is the stochastic discount factor (i.e., the intertemporal marginal rate of substitution) and assume that the price is bounded. (68) means that *the stock price is the sum of all expected discounted future dividends*.
- Example: If the utility function displays *risk neutrality* (i.e., $u_1(\cdot)$ is constant),

$$p_s = E_s \sum_{j=1}^{\infty} \left[\beta^j y_{s+j} \right] = E_s \sum_{j=1}^{\infty} \left[\frac{y_{s+j}}{\left(1 + r_f\right)^j} \right], \tag{69}$$

  which means that the stock price is the sum of expected future dividends discounted at the constant risk free rate.

- The difference between (68) and (69) is the necessity of discounting the flow of expected dividends at a rate higher than the risk free rate, so as to include *a risk premium*. Which factors affect risk premium is a central issue in finance.
- Another examples: If the utility function is log, the price is

$$p_s = E_s \sum_{j=1}^{\infty} \left(\beta^j \frac{y_s}{y_{s+j}} y_{s+j} \right) = E_s \sum_{j=1}^{\infty} \left(\beta^j y_s \right) \quad (70)$$

$$= \frac{\beta}{1-\beta} y_s, \quad (71)$$

Calculating the equilibrium price function.

- The Euler equation, $u_1(y_s)p_s = \beta E_s\left[u_1(y_{s+1})\left(y_{s+1}+p_{s+1}\right)\right]$, *implicitly* defines the equilibrium price. And we can calculate the actual equilibrium prices once specifying parameter values and function forms: select β, the utility function $u(c)$, and the transition matrix Π.
- Specifically, we can solve for $\left\{p(y^j), j = 1, \cdots, N\right\}$ as the solution to a system of linear equations:

$$u_1(y^1)p\left(y^1\right) = \beta\sum_{j=1}^{N}\pi_{1j}\left[u_1(y^j)\left(y^j+p\left(y^j\right)\right)\right] \tag{72}$$

$$\cdots \tag{73}$$

$$u_1(y^N)p\left(y^N\right) = \beta\sum_{j=1}^{N}\pi_{Nj}\left[u_1(y^j)\left(y^j+p\left(y^j\right)\right)\right] \tag{74}$$

with *unknowns* $\left\{p(y^j), j = 1, \cdots, N\right\}$.

Numerica Example

- Suppose that $\beta = 0.96$, $u(c) = \ln(c)$, $(y^1, y^2, y^3) = (1.5, 1, 0.5)$, and the transition matrix Π :.

	Three-state probability transition matrix
	output in $s+1$ (y_{s+1})
output in s (y_s)	$\Pi = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$
where $\pi_{ij} = prob\left(y_{s+1} = y^j \middle| y_s = y^i\right)$ for any s.	

- Hence, we have three equations and three unknowns $\left\{p(y^j), j = 1, 2, 3\right\}$ and can solve for the equilibrium prices:

$$p(1) = 24; p(1.5) = 36; p(0.5) = 12. \tag{75}$$

Link Asset Prices to Asset Returns

- Define the return of security j from period t to $t+1$ as

$$1 + r_{j,t+1} = \frac{p_{j,t+1} + y_{j,t+1}}{p_{j,t}}. \tag{76}$$

Using this definition, the above Euler equation can be rewritten as

$$1 = \beta E_t \left[\frac{u_1(c_{t+1})}{u_1(c_t)} (1 + r_{j,t+1}) \right] \tag{77}$$

- Let q_t denote the price in t of a one-period risk free bond in zero net supply, which pays *one unit* of consumption in every state in $t+1$. Hence,

$$q_t u_1(c_t) = \beta E_t [u_1(c_{t+1}) \cdot 1], \tag{78}$$

where q_t is the *equilibrium* price at which the agent desires to hold zero units of the security, and thus supply equals demand.

- Since $(1 + r_{f,t+1}) q_t = 1$, we have

$$\frac{1}{1 + r_{f,t+1}} = q_t = \beta E_t \left[\frac{u_1(c_{t+1})}{u_1(c_t)} \right], \tag{79}$$

which means that in the risk neutrality case, the risk free rate must be a constant.

- Combining the above two pricing equations, we have

$$1 = \beta E_t \left[\frac{u_1(c_{t+1})}{u_1(c_t)} \right] E_t [1 + r_{j,t+1}] + \beta \operatorname{covar}_t \left[\frac{u_1(c_{t+1})}{u_1(c_t)}, 1 + r_{j,t+1} \right], \tag{80}$$

where we use the fact that for two random variables: $\operatorname{covar}[x, y] = E[xy] - E[x] E[y]$.

- Denote $E_t [1 + r_{j,t+1}] = 1 + \overline{r}_{j,t+1}$, we have

$$1 = \frac{1 + \overline{r}_{j,t+1}}{1 + r_{f,t+1}} + \beta \operatorname{covar}_t \left[\frac{u_1(c_{t+1})}{u_1(c_t)}, r_{j,t+1} \right] \tag{81}$$

- Rearranging gives

$$\bar{r}_{j,t+1} - r_{f,t+1} = -\beta\left(1 + r_{f,t+1}\right) \operatorname{covar}_t \left[\frac{u_1\left(c_{t+1}\right)}{u_1\left(c_t\right)}, r_{j,t+1}\right] \quad (82)$$

- This equation is the central relationship of the CCAPM and is very similar to the pricing equation obtained in the last lecture. The LHS is *the risk premium* on security j.
    - It means that the risk premium will be large when $\operatorname{covar}_t\left[\frac{u_1(c_{t+1})}{u_1(c_t)}, r_{j,t+1}\right]$ is large and negative, that is, for those securities paying high returns when consumption is high (i.e., $u_1(\cdot)$ is low), and low returns when consumption is high (i.e., $u_1(\cdot)$ is high).
    - These securities are not very desirable for reducing consumption risk because they pay high returns when investors don't need them and low returns when they are most needed.
    - Since they are not desirable, they have a *low* price and *high* expected returns for compensation.

A Discussion

- The standard CAPM in finance tells us a security is relatively undesirable and thus commands a high return when it covaries positively with the market portfolio. The Consumption CAPM adds some some further degree of precision: from the viewpoint of consumption smoothness and risk diversification, an asset is desirable if it has a high return when consumption is low and vice versa.
- C-CAPM is more convincing in the multiperiod context because the value of an asset is to provide the investor intermediate consumption over time; consequently, the key to an asset's value is its covariation with *the marginal utility of consumption*.
- An unappealing feature of the above CCAPM is that the marginal utility of consumption is not observable. We can eliminate this feature by adopting a specific utility function.

- So far, we have assumed that output (=dividend=consumption in equilibrium) is stationary. In reality, consumption and output are growing over time. Here we assume that output growth rather than output itself follows a distribution with possible finite states: $(x_1, \cdots, x_N)$ whose realizations are governed by a stochastic process with transition matrix Π. Then for whatever x_i is realized in period $t+1$:
$$d_{t+1} = x_{t+1} y_t = x_{t+1} c_t = x_i c_t. \tag{83}$$
- Using the CRRA utility, in equilibrium ($y_t = c_t$):
$$y_t^{-\gamma} p(y_t, x_i) = \beta \sum_{j=1}^{N} \pi_{ij} (x_j y_t)^{-\gamma} [x_j y_t + p(x_j y_t, x_j)] \text{ or} \tag{84}$$
$$p(y_t, x_i) = \beta \sum_{j=1}^{N} \pi_{ij} (x_j)^{-\gamma} [x_j y_t + p(x_j y_t, x_j)], \tag{85}$$
which means that the SDF is determined exclusively by the consumption growth rate x_j.

- Just like Mehra and Prescott (1985), we guess

$$p(y_t, x_i) = v_i y_t, \tag{86}$$

for a set of constants $\{v_1, \cdots, v_N\}$, each should be identified with the corresponding growth rate.

- With this functional form, the asset pricing equation reduces to

$$v_i y_t = \beta \sum_{j=1}^{N} \pi_{ij} (x_j)^{-\gamma} [x_j y_t + v_j x_j y_t] \text{ or} \tag{87}$$

$$v_i = \beta \sum_{j=1}^{N} \pi_{ij} (x_j)^{-\gamma} [x_j + v_j x_j] = \beta \sum_{j=1}^{N} \pi_{ij} (x_j)^{1-\gamma} (1 + v_j), \tag{88}$$

which is again a system of linear equations in the N unknowns $\{v_1, \cdots, v_N\}$.

- Thus, for any state $(y, x_j) = (c, x_j)$, the equilibrium asset price is

$$p(y_t, x_j) = v_j y_t, \tag{89}$$

if we assume that the current state is (y, x_i) while next period it is $(x_j y, x_j)$, then the one-period return is

$$r_{ij} = \frac{p(x_j y_t, x_j) + x_j y - p(y, x_i)}{p(y, x_i)} \tag{90}$$

$$= \frac{v_j x_j y_t + x_j y - v_i y}{v_i y_t} = \frac{x_j (1 + v_j)}{v_i} - 1, \tag{91}$$

- Hence, the mean or expected return, conditional on state i, is $r_i = \sum_{j=1}^{N} \pi_{ij} r_{ij}$,and the unconditional equity return is given by $E[r] = \sum_{j=1}^{N} \overline{\pi}_j r_j$, where $\overline{\pi}_j$ are long-run stationary probability of each state.
- The price of the risk free asset is

$$p^f(c, x_i) = \beta \sum_{j=1}^{N} \pi_{ij} (x_j)^{-\gamma}. \tag{92}$$

The Empirical Validity of the CCAPM

- A few key empirical observations regarding financial returns in US markets:

$$E\left[R_{t+1}^{j}\right] - R^{f} = 7\%,$$

  based on Campbell (2003)'s U.S. data from 1947.2 – 1998.4.

- The equity premium puzzle found by Mehra and Prescott (1985): the standard CCAPM is completely unable to replicate the high observed equity premium once reasonable parameter values are inserted in the model.

The Reasoning of the Puzzle

- According to the CCAPM theory, the only factors determining the characteristics of asset returns are the RA's utility function, the subjective discount factor, and the consumption process.
  - The utility function: CRRA $\frac{c^{1-\gamma}}{1-\gamma}$ and empirical studies have placed γ in the range of $[1, 5)$.
  - The consumption process. In reality, consumption is growing over time. If there were no uncertainty in the model, and if the constant growth rate of consumption were to equal to its long-run historical average (around 1.0183), the asset pricing equation would reduce to

$$1 = \beta E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t+1} \right] = \beta g^{-\gamma} R, \tag{93}$$

where R_{t+1} is the gross rate on capital, g and R are historical averages of consumption growth and the gross rate.

- For $\gamma = 1, g = 1.0183$, and $R = 1.04$, we can solve for the implied $\beta \simeq 0.97$. (Note that the economywide debt-to-equity ratios are not very different from 1.) Since we use an annual estimate for g, the resulting β must be viewed as an annual or yearly subjective discount factor. Similarly, on quarterly basis, $\beta = 0.99$.
    - If assume that $\gamma = 2$, the implied $\beta = 0.99$ annually and a quarterly β even closer to 1. Specifically, assuming higher rates of risk aversion would be *incompatible* with maintaining the hypothesis of a time discount factor $\beta < 1$.
    - At the root of this difficulty is the low return on the risk free asset (1%). Highly risk averse individuals want to smooth consumption over time ($1/\gamma$ low if γ high), meaning they want to transfer consumption from good times to bad times. Hence, when consumption is growing predictably, the good times lies in the future. Agents want to borrow now against their future income.
    - In the RA model, it is hard to reconcile with a low rate on borrowing: everyone is on the same side of the market and then inevitably forces a higher rate. As a result, we need either an independent explanation for the low average risk free rate or accept $\beta > 1$. Here, we just limit $\gamma \leq 2$.

A Way to Test the CCAPM: Hansen-Jagannathan (HJ) Bounds

- The bound proposed by HJ (1991) leads to a falsification of the standard CCAPM and can also be applied in other asset pricing formulations.
- For all homogeneous agent economies, the equilibrium asset pricing can be rewritten as:

$$p(s_t) = E_t\left[m_{t+1}(s_{t+1})X(s_{t+1})\right], \tag{94}$$

where s_t is the state today, $X(s_{t+1})$ is the total return in next period, and $m_{t+1}(s_{t+1})$ is the SDF:

$$m_{t+1}(s_{t+1}) = \beta\frac{u_1(c_{t+1}(s_{t+1}))}{u_1(c_t)}. \tag{95}$$

- Suppress the state dependence, the pricing equation:

$$p_t = E_t\left[m_{t+1}X_{t+1}\right] \Longleftrightarrow 1 = E_t\left[m_{t+1}R_{t+1}\right], \tag{96}$$

where R_{t+1} is the gross return.

- Since the equation holds for each state s_t, it also holds unconditionally:

$$1 = E\left[mR\right], \tag{97}$$

where E denotes the unconditional expectation. For any two assets, i and j,

$$E\left[m(R_i - R_j)\right] = 0, \text{ or } E\left[mR_{i-j}\right] = 0, \tag{98}$$

which implies that

$$\begin{aligned} E\left[m\right]E[R_{i-j}] + \mathrm{covar}\left[m, R_{i-j}\right] &= 0 \Longrightarrow \quad (99) \\ E\left[m\right]E[R_{i-j}] + \rho\left[m, R_{i-j}\right]\mathrm{sd}\left[m\right]\mathrm{sd}\left[R_{i-j}\right] &= 0 \Longrightarrow \\ \frac{E\left[R_{i-j}\right]}{\mathrm{sd}\left[R_{i-j}\right]} + \rho\left[m, R_{i-j}\right]\frac{\mathrm{sd}\left[m\right]}{E\left[m\right]} &= 0. \quad (100) \end{aligned}$$

- Since $\rho\left[m, R_{i-j}\right] \leq 1$,

$$\frac{\mathrm{sd}\left[m\right]}{E\left[m\right]} \geq \frac{\left|E\left[R_{i-j}\right]\right|}{\mathrm{sd}\left[R_{i-j}\right]} \tag{101}$$

- The inequality is referred as *the Hansen-Jagannathan lower bound* on the SDF. If i is the market portfolio and j is the risk free asset, we have:

$$\frac{\text{std}[m]}{E[m]} \geq \frac{|E[R_M - R_f]|}{\text{std}[R_{M-f}]} = \frac{|E[R_M - R_f]|}{\text{std}[R_M]} = \frac{0.062}{0.167} = 0.37. \quad (102)$$

- We now can check whether this bound is satisfied for the standard CCAPM in which $m = \beta(x_t)^{-\gamma}$. Given

$$E[m] = \beta \exp\left(-\gamma\mu_x + \frac{1}{2}\gamma^2\sigma_x^2\right) = 0.96 \text{ if } \gamma = 2, \quad (103)$$

  for the HJ bound to be satisfied, the standard deviation of the SDF cannot be much lower than

$$0.37 \cdot 0.96 = 0.355. \quad (104)$$

- Given the information about x (lognormal distribution), it is straightforward to compute that

$$\text{std}\left[m\right] = 0.002! \tag{105}$$

which is much lower than what is required for reaching the HJ bound (0.355).

- Intuition: aggregate consumption is just too smooth and the MU of consumption doesn't vary sufficiently to satisfy the HJ bound implied by asset data.

Summary

- Reviewing the source of the failure of CCAPM in matching the data. Recall the original pricing equation

$$
\begin{aligned}
r_{M,t+1} - r_{f,t+1} &= -\beta\left(1 + r_{f,t+1}\right) \text{covar}_{\,t}\left[\frac{u_1\left(c_{t+1}\right)}{u_1\left(c_t\right)}, r_{M,t+1}\right] \\
&= -\left(1 + r_{f,t+1}\right)\rho\left[\beta\frac{u_1\left(c_{t+1}\right)}{u_1\left(c_t\right)}, r_{M,t+1}\right] \text{std}\left[m\right] \text{std}\left[R\right.
\end{aligned}
$$

- Implications: the equity premium depends on
  - The standard deviation of the SDF
  - The standard deviation of the market portfolio
  - The correlation between the two variables.
- Hence, for the US and other industrial countries, the problem with the CCAPMs is that aggregate consumption does not vary much at all. To make this model better fit the data, we must modify it in a way that will increase *the standard deviation of the relevant SDF*.

Habit Formation

- Main objective: admit utility functions that exhibit higher rates of risk aversion and thus can translate small variations in consumption into a large variability of the SDF.
- One way to achieve this objective without causing the risk free rate puzzle (which is exacerbated if we simply assume a higher RRA γ): introducing some form of habit formation.
- *Habit formation:* the agent's utility today is determined not by her *absolute* consumption level, but rather by the *relative* position of her current consumption. The stock of habit can summarizes either *her past consumption* history (with more or less weight placed on distant or old consumption levels) or the history of *aggregate/average consumption* (summarizing in a sense the consumption habits of her neighbors: a "keeping up with the Joneses" effect).
- Utility of consumption is primarily dependent on departures from prior consumption history, either one's own or that of a social reference group.

Habit Formation and The Equity Premium Puzzle

- The RA's preference takes the following form:

$$u(c_t, c_{t-1}) = u(c_t - \chi c_{t-1}) = \frac{(c_t - \chi c_{t-1})^{1-\gamma}}{1-\gamma}, \tag{106}$$

  where $\chi \leq 1$ is a parameter. When $\chi = 1$, the utility depends only on the deviation of current consumption c_t from the previous period's consumption c_{t-1}. Note that a general specification of habit formation can be written as

$$u(c_t, x_t) = \frac{(c_t - \chi x_t)^{1-\gamma}}{1-\gamma}, \tag{107}$$

  where

$$x_t = (1-\theta)\, x_{t-1} + c_{t-1}. \tag{108}$$

- Actual data indicate that aggregate consumption in the US and other developed countries is *very smooth.* This implies that $(c_t - c_{t-1})$ is likely to be very small most of the time.

- For this specification, the agent's *effective* relative risk aversion reduces to

$$R_R(c_t) = -\frac{c_t u''(\cdot)}{u'(\cdot)} = \frac{\gamma}{s_t}, \tag{109}$$

where

$$s_t = 1 - \chi(c_{t-1}/c_t). \tag{110}$$

With $c_{t-1} \approx c_t$, the degree of effective risk aversion $R_R(c_t)$ could thus be very high, even with a low γ, and the RA will appear as though he is very risk averse. This opens the possibility for a very high return on the risky asset.

- Note that when $s_t \downarrow 0$, $R_R(c_t) \uparrow \infty$. With habit, the SDF can be written as

$$\text{SDF} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \left(\frac{s_{t+1}}{s_t}\right)^{-\gamma}, \tag{111}$$

where $\left(\frac{s_{t+1}}{s_t}\right)^{-\gamma}$ could be very volatile and correlated. If so, habit formation might help explain the equity premium puzzle. See Campbell and Cochrane (JPE1999) for details.

Epstein-Zin Recursive Utility

- Epstein and Zin (1989, 1991) propose a class of utility functions that allows each dimension to be parameterized *separately*. Specifically, preferences are defined recursively over current (known) consumption and a certainty equivalent of the next period's total utility:

$$U_t = U(c_t, c_{t+1}, c_{t+2,} \cdots) = W(c_t, R_t(U_{t+1})), \tag{112}$$

  where $R_t(U_{t+1}) \triangleq CE_t$ denotes the certainty equivalence in terms of period-t consumption of the uncertain total utility in the future periods. Consider the CES aggregator function:

$$U(c_t, CE_t) = \left[(1-\beta)c_t^{1-1/\rho} + \beta\left(CE_t\right)^{1-1/\rho}\right]^{\frac{1}{1-1/\rho}}. \tag{113}$$

- $$R_t(U_{t+1}) = G^{-1}\left(E_t\left[G\left(U_{t+1}\right)\right]\right), \tag{114}$$

  where W and G are increasing and concave.

Asset Pricing Implications

- Consider a CRRA specification for the certainty equivalent (CE). For a random variable U_{t+1}, the period-$t+1$ onward utility, the certainty equivalent $CE_t(U_{t+1})$ $(= CE_t)$ is defined as follows:

$$
\begin{aligned}
\left[CE_t(U_{t+1})\right]^{1-\gamma} &= E_t\left[U_{t+1}^{1-\gamma}\right] \text{ when } \gamma > 0 \text{ and } \gamma \neq 1, && (115) \\
\implies & CE_t(U_{t+1}) = \left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{1/(1-\gamma)} && (116)
\end{aligned}
$$

- (113) can be rewritten as:

$$
U(c_t, CE_t) = \left[(1-\beta)c_t^{\frac{1-\gamma}{\theta}} + \beta\left(CE_t\right)^{\frac{1-\gamma}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \qquad (117)
$$

where $\gamma, \rho > 0, \gamma \neq 1,\ \theta = \frac{1-\gamma}{1-1/\rho}$.

- If $\gamma = 1/\rho$ (i.e., $\theta = 1$) or if consumption is deterministic, we have the usual standard time-separable expected utility setting with the discount factor β and IES ρ (RRA $\gamma = 1/\rho$).

- (Conti.) A proof. When $\gamma = 1/\rho$ ($\theta = 1$),

$$
\begin{aligned}
U_t(c_t, CE_t)^{1-\gamma} &= (1-\beta)c_t^{1-\gamma} + \beta\left(CE_t\right)^{1-\gamma} \\
U_t^{1-\gamma} &= (1-\beta)c_t^{1-\gamma} + \beta\left(\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{1/(1-\gamma)}\right)^{1-\gamma} \\
U_t^{1-\gamma} &= (1-\beta)c_t^{1-\gamma} + \beta E_t\left[U_{t+1}^{1-\gamma}\right].
\end{aligned}
$$

- By unwinding the recursion, we get

$$
\begin{aligned}
U_t^{1-\gamma} &= (1-\beta)c_t^{1-\gamma} + \beta E_t\left[(1-\beta)c_{t+1}^{1-\gamma} + \beta E_t\left[U_{t+2}^{1-\gamma}\right]\right] \\
&= \cdots = (1-\beta)E_t\left[\sum_{s=t}^{\infty}\beta^{s-t}c_s^{1-\gamma}\right],
\end{aligned}
$$

which reduces to $U_t = \left[(1-\beta)E_t\left[\sum_{s=t}^{\infty}\beta^{s-t}c_s^{1-\gamma}\right]\right]^{1/(1-\gamma)}$ and is equivalent to the standard EU: $V_t = E_t\left[\sum_{j=0}^{\infty}\beta^j \frac{c_{t+j}^{1-\gamma}}{1-\gamma}\right]$. (Note that $V_t = \frac{1}{(1-\beta)(1-\gamma)}U_t^{1-\gamma}$ is an increasing function of U_t and therefore represents the same preference as U_t.)

Asset Pricing Implications of RU

- Epstein and Zin (1989,1991): derived the following asset pricing equation:

$$E_t \left\{ \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-1/\rho} \right]^{\theta} \left[\frac{1}{R^p_{t+1}} \right]^{1-\theta} R^j_{t+1} \right\} = 1, \qquad (118)$$

  where R^p_{t+1} is the period-$(t+1)$ return on the market portfolio, and R^j_{t+1} is the return on some asset in it. Note that when $\theta = 1$ $(\gamma = 1/\rho)$, the above pricing equation reduces to the standard time-separable CCAPM case.

- Note that in the EZ model, the SDF is a geometric average (with weights θ and $1-\theta$) of the SDF of the standard CCAPM $(\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma})$ and the SDF of the log case $(\frac{1}{R^p_{t+1}})$.

- (Conti.) Hence, two covariances matter for an asset's return pattern:
  1. the covariance of asset returns with consumption growth;
  2. the covariance of asset return with the return on the market portfolio.
- The covariance with consumption growth captures its risk across successive time periods (*intertemporally*, as in the standard CCAPM, while the covariance with the market portfolio captures its *atemporal* systematic risk (as in the standard static CAPM).

Proof.

We first rewrite the definition of the recursive utility together with the certainty equivalent (115) as follows:

$$U_t^{1-\gamma} = \left[(1-\beta) c_t^{1-1/\rho} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{(1-1/\rho)/(1-\gamma)} \right]^{\frac{1-\gamma}{1-1/\rho}}, \quad (119)$$

which can be rewritten as

$$\begin{aligned} U_t &= \left[(1-\beta) c_t^{1-1/\rho} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{(1-1/\rho)/(1-\gamma)} \right]^{\frac{1}{1-1/\rho}} \quad (120) \\ &= \left[(1-\beta) c_t^{1-1/\rho} + \beta \left(CE_t \right)^{1-1/\rho} \right]^{\frac{1}{1-1/\rho}} \equiv f\left(c_t, CE_t\right). \end{aligned}$$

Further, the budget constraint is

$$A_{t+1} = R_{t+1}^p \left(A_t - c_t \right), \text{ for any } t. \quad (121)$$

□

Proof.

(Conti.) With the RU, the SDF can be written as

$$SDF_{t,t+1} = \frac{\partial U_t / \partial c_{t+1}}{\partial U_t / \partial c_t} = f_{CE}\left(c_t, CE_t\right)\left(\frac{U_{t+1}}{CE_t}\right)^{-\gamma}\frac{f_c\left(c_{t+1}, CE_{t+1}\right)}{f_c\left(c_t, CE_t\right)},$$

where the RHS is evaluated at the optimal consumption process,

$$\begin{aligned}
\frac{\partial U_t}{\partial c_t} &= f_c\left(c_t, CE_t\right), \\
\frac{\partial U_t}{\partial c_{t+1}} &= f_{CE}\left(c_t, CE_t\right)\frac{d\left(CE_t\right)}{d\left(U_{t+1}\right)}\frac{\partial U_{t+1}}{\partial c_{t+1}} \\
&= f_{CE}\left(c_t, CE_t\right) CE_t^{\gamma} U_{t+1}^{-\gamma} f_c\left(c_{t+1}, CE_{t+1}\right)
\end{aligned}$$

where we use the fact that

$$\frac{d\left(CE_t\right)}{d\left(U_{t+1}\right)} = \left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}-1} U_{t+1}^{-\gamma} = CE_t^{\gamma} U_{t+1}^{-\gamma}.$$

□

Proof.

(Conti.) Note that given (120), the derivatives are

$$
\begin{aligned}
f_c\left(c_t, CE_t\right) &= (1-\beta)c_t^{-1/\rho}U_t^{1/\rho}, \\
f_{CE}\left(c_t, CE_t\right) &= \beta\left(CE_t\right)^{-1/\rho}U_t^{1/\rho}, \\
f_c\left(c_{t+1}, CE_{t+1}\right) &= (1-\beta)c_{t+1}^{-1/\rho}U_{t+1}^{1/\rho},
\end{aligned}
$$

and the SDF can be reduced to

$$
\begin{aligned}
SDF_{t,t+1} &= \beta\left(CE_t\right)^{-1/\rho}U_t^{1/\rho}\left(\frac{U_{t+1}}{CE_t}\right)^{-\gamma}\frac{(1-\beta)c_{t+1}^{-1/\rho}U_{t+1}^{1/\rho}}{(1-\beta)c_t^{-1/\rho}U_t^{1/\rho}} \\
&= \beta\left(\frac{U_{t+1}}{CE_t}\right)^{1/\rho-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho}. \qquad (122)
\end{aligned}
$$

□

- In reality, however, we cannot observe the future utility index, $\frac{U_{t+1}}{CE_t}$. In the next step, we will link the SDF to the return to the market portfolio.

Link the SDF to the Return to the Market Portfolio

- First, given $A_{t+1} = R^p_{t+1}(A_t - c_t)$. and R^p_{t+1} satisfying
$$1 = E_t\left[SDF_{t,t+1}R^p_{t+1}\right], \tag{123}$$
we have
$$A_t = c_t + E_t\left[SDF_{t,t+1}A_{t+1}\right]. \tag{124}$$
- Second, we will show that at optimum, we have
$$A_t = \frac{J_t}{f_c(c_t, CE_t)},$$
where the value function J_t satisfying
$$J_t = \max_{c,\alpha} U_t = \max_{c,\alpha}\left[(1-\beta)c_t^{1-1/\rho} + \beta\,(CE_t)^{1-1/\rho}\right]^{\frac{1}{1-1/\rho}}. \tag{125}$$
We guess that $A_t = \frac{J_t}{f_c(c_t, CE_t)}$ holds, i.e.,
$$A_t = \frac{J_t}{f_c(c_t, CE_t)} = \frac{J_t}{(1-\beta)c_t^{-1/\rho}J_t^{1/\rho}} = \frac{1}{1-\beta}c_t^{1/\rho}J_t^{1-1/\rho}.$$

- (Conti.) Substituting this into (123), we have

$$\begin{aligned}
& E_t\left[SDF_{t,t+1}A_{t+1}\right] \\
= & E_t\left[\beta\left(\frac{U_{t+1}}{CE_t}\right)^{1/\rho-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho}\frac{1}{1-\beta}c_{t+1}^{1/\rho}J_{t+1}^{1-1/\rho}\right] \\
= & \frac{\beta}{1-\beta}c_t^{1/\rho}E_t\left[\left(\frac{1}{CE_t}\right)^{1/\rho-\gamma}J_{t+1}^{1-\gamma}\right] \\
= & \frac{\beta}{1-\beta}c_t^{1/\rho}\left(CE_t\right)^{1-1/\rho}
\end{aligned}$$

- Substituting this equation into (124):

$$\begin{aligned}
\frac{1}{1-\beta}c_t^{1/\rho}J_t^{1-1/\rho} &= c_t+\frac{\beta}{1-\beta}c_t^{1/\rho}\left(CE_t\right)^{1-1/\rho}\Longrightarrow \\
J_t^{1-1/\rho} &= \left(1-\beta\right)c_t^{1-1/\rho}+\beta\left(CE_t\right)^{1-1/\rho},
\end{aligned}$$

which is just (125).

- (Conti.) The return on wealth can be written as

$$
\begin{aligned}
R_{t+1}^{p} &= \frac{A_{t+1}}{A_t - c_t} = \frac{A_{t+1}}{E_t\left[SDF_{t,t+1}A_{t+1}\right]} = \frac{\frac{1}{1-\beta}c_{t+1}^{1/\rho}J_{t+1}^{1-1/\rho}}{\frac{\beta}{1-\beta}c_t^{1/\rho}\left(CE_t\right)^{1-1/\rho}} \\
&= \frac{c_{t+1}^{1/\rho}J_{t+1}^{1-1/\rho}}{\beta c_t^{1/\rho}\left(CE_t\right)^{1-1/\rho}} = \beta^{-1}\left(\frac{c_{t+1}}{c_t}\right)^{1/\rho}\left(\frac{J_{t+1}}{CE_t}\right)^{1-1/\rho}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
SDF_{t,t+1} &= \beta\left(\frac{J_{t+1}}{CE_t}\right)^{1/\rho-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho} \\
&= \beta\left[\beta R_{t+1}^{p}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho}\right]^{\frac{1/\rho-\gamma}{1-1/\rho}}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho} \\
&= \beta\left[\beta R_{t+1}^{p}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho}\right]^{\theta-1}\left(\frac{c_{t+1}}{c_t}\right)^{-1/\rho} \\
&= \beta^{\theta}\left(R_{t+1}^{p}\right)^{\theta-1}\left(\frac{c_{t+1}}{c_t}\right)^{-\theta/\rho}
\end{aligned}
$$

Implications

- If we assume that consumption growth and the return to the risky asset are jointly normally distributed, (118) can be written in the following log-linear form:

$$E_t\left[r^j_{t+1}\right] - r^f + \frac{\omega_j^2}{2} = \frac{\theta}{\rho}\operatorname{cov}_t\left(r^j_{t+1}, \Delta c_{t+1}\right) + (1-\theta)\operatorname{cov}_t\left(r^j_{t+1}, r^p_{t+1}\right) \quad (126)$$

  which means that the expected excess return on the risky asset is a weighted average of the risky asset's covariance with consumption growth (divided by the IES ρ) and the asset's covariance with the market portfolio return.