ECON2285: Mathematical Economics

Yulei Luo
SEF of HKU

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Course Outline

- *Economics*: The study of the choices people (consumers, firm managers, and governments) make to attain their goals, given their scarce resources.
- *Economic model*: Simplified version of reality used to analyze real-world economic situations.

This course will mainly focus on how to use *mathematical methods* to solve *economic models*.

After solving the models, you may use economic data to test the hypothesis derived from models. If you are interested in testing economic models, you may take Econ0701: *Econometrics*.
Course Outline

- **Static Analysis**
  - Main objective: characterize partial and general equilibrium: \( \text{demand} = \text{supply} \) and solve for equilibrium prices and quantities.
  - Mathematical tools: matrix algebra
  - Economic applications: market equilibrium models and national-income models

- **Comparative-Static Analysis**
  - Main objective: compare different equilibrium states that are associated with different sets of values of parameters or exogenous variables (i.e., they are given and not determined by the model structure).
  - Mathematical tools: calculus (differentiation, partial differentiation, etc.)
  - Economic applications: market equilibrium models and national-income models
(Continued.) Static Optimization Analysis

- Main objective: characterize the optimizing behavior of consumers and firms behind equilibrium at one period (static, atemporally).
- Mathematical tools: unconstrained and constrained static optimization
- Economic applications: utility maximization, profit maximization, and cost minimization.

Dynamic Analysis

- Main objective: characterize the evolution of economic variables over time (dynamic, intertemporally).
- Mathematical tools: first-order and higher-order difference (discrete-time) or differential (continuous-time) equations.
- Economic applications: economic growth models, cobweb model, etc.

Dynamic Optimization

- Main objective: characterize the optimal behavior of agents over time.
- Mathematical tools: optimal control theory (the Lagrange multiplier method) or dynamic programming (The Bellman equation method)
- Economic applications: optimal economic growth models, life-cycle optimal consumption-saving models, optimal investment models.
In essence, an equilibrium for a specific model is a situation that is characterized by a lack of tendency to change. It is for this reason that the analysis of equilibrium is referred to as statics.

We will discuss two examples of equilibrium. One is the equilibrium attained by a market under given demand and supply conditions. The other is the equilibrium of national income under given conditions of total consumption and investment patterns.
Partial equilibrium model (linear)

- In a static equilibrium model, the standard problem is to find the set of values of the endogenous variables (i.e., they are *not given* and are determined by the model structure) which will satisfy the equilibrium conditions of the model.

- Partial equilibrium market model: a model of price determination in a single market. Three variables:

  \[ Q_d = \text{the quantity demanded of the commodity} \]
  \[ Q_s = \text{the quantity supplied of the commodity} \]
  \[ P = \text{the price of commodity}. \]

  And the equilibrium condition is

  \[ Q_d = Q_s. \]  

- The model setup is then

  \[ Q_d = Q_s \]  
  \[ Q_d = a - bP \]  
  \[ Q_s = -c + dP \]

  where \(a, b, c,\) and \(d\) are all positive.
One way to find the equilibrium is by successive elimination of variables and equations through substitution. From $Q_d = Q_s$, we have

$$a - bP = -c + dP.$$  \hspace{1cm} (5)

Since $b + d \neq 0$, the equilibrium price is then

$$P^* = \frac{a + c}{b + d},$$ \hspace{1cm} (6)

and the equilibrium quantity can be obtained by substituting $P^*$ into either $Q_d$ or $Q_s$:

$$Q^* = \frac{ad - bc}{b + d},$$ \hspace{1cm} (7)

where we assume that $ad - bc > 0$ to make the model economically meaningful.
The partial equilibrium market model can be nonlinear. The model setup is then

\[ Q_d = Q_s \tag{8} \]
\[ Q_d = 4 - P^2 \tag{9} \]
\[ Q_s = -1 + 4P \tag{10} \]

As before, the three-equation system can be reduced to a single equation by substitution:

\[ P^2 + 4P - 5 = 0, \tag{11} \]

which can be solved by applying the quadratic formula:

\[ P^* = 1 \text{ or } -5. \tag{12} \]

Note that only the first root is economically meaningful.
In the above, we only consider a single market. In the real world, there would normally exist many goods. Thus, a more realistic model for the demand and supply functions of a commodity should take into account the effects not only of the price of the commodity itself but also of the prices of other commodities.

As a result, the price and quantity variables of multiple commodities must enter endogenously into the model. Consequently, the equilibrium condition of an \( n \)-commodity market model will involve \( n \) equations, one for each commodity, in the form

\[
E_i = Q_{i,d} - Q_{i,s} = 0, \tag{13}
\]

where \( i = 1, \ldots, n \),

\[
Q_{i,d} = Q_{i,d}(P_1, \ldots, P_n) \quad \text{and} \quad Q_{i,s} = Q_{i,s}(P_1, \ldots, P_n) \tag{14}
\]

are the demand and supply functions of commodity \( i \). Thus solving \( n \)-equation system for \( P \):

\[
E_i(P_1, \ldots, P_n) = 0 \tag{15}
\]

if a solution does exist.
Two commodity market model

The model setup is as follows

\[ Q_{1,d} = Q_{1,s} \quad (16) \]
\[ Q_{1,d} = a_0 + a_1 P_1 + a_2 P_2 \quad (17) \]
\[ Q_{1,s} = b_0 + b_1 P_1 + b_2 P_2 \quad (18) \]
\[ Q_{2,d} = Q_{2,s} \quad (19) \]
\[ Q_{2,d} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \quad (20) \]
\[ Q_{2,s} = \beta_0 + \beta_1 P_1 + \beta_2 P_2 \quad (21) \]

which implies that

\[ (a_0 - b_0) + (a_1 - b_1) P_1 + (a_2 - b_2) P_2 = 0 \quad (22) \]
\[ (\alpha_0 - \beta_0) + (\alpha_1 - \beta_1) P_1 + (\alpha_2 - \beta_2) P_2 = 0 \quad (23) \]
(Continued.) Denote that

\[ c_i = a_i - b_i; \quad d_i = \alpha_i - \beta_i, \]  

(24)

where \( i = 0, 1, 2 \). The above linear equations can be written as

\[ c_1 P_1 + c_2 P_2 = -c_0 \]  
\[ d_1 P_1 + d_2 P_2 = -d_0, \]  

(25) \hspace{1cm} (26)

which can be solved by further elimination of variables:

\[ P_1^* = \frac{c_2 d_0 - c_0 d_2}{c_1 d_2 - c_2 d_1}; \quad P_2^* = \frac{c_0 d_1 - c_1 d_0}{c_1 d_2 - c_2 d_1}. \]  

(27)

Note that for the \( n \)-commodity linear market model, we have \( n \) linear equations and \( n \) unknowns. However, an equal number of equations and unknowns doesn’t necessarily guarantee the existence of a unique solution. Only when the number of unknowns equals to the number of functional independent equations, the solution exists and is unique. We thus need systematic methods to test the existence of a unique solution.
Consider a simple Keynesian national-income model,

\[ Y = C + I_0 + G_0 \] (Equilibrium condition) \hspace{1cm} (28)

\[ C = a + bY \] (The consumption function), \hspace{1cm} (29)

where \( Y \) and \( C \) stand for the endogenous variables national income and consumption expenditure, respectively, and \( I_0 \) and \( G_0 \) are the exogenously determined investment and government spending. Following the same procedure, the equilibrium income and consumption can be solved as follows.

\[ Y^* = \frac{a + I_0 + G_0}{1 - b} \] \hspace{1cm} (30)

\[ C^* = \frac{a + b(I_0 + G_0)}{1 - b} \] \hspace{1cm} (31)
Why matrix algebra?

- As more and more commodities are incorporated into the model, it becomes more and more difficult to solve the model by substitution. A method suitable for handling a large system of simultaneous equations is matrix algebra.
- Matrix algebra provides a compact way of writing an equation system; it leads to a way to test the existence of a solution by evaluation of a determinant – a concept closely related to that of a matrix; it also gives a method to find that solution if it exists.
**Quick review of matrix algebra**

- **Matrix:** a matrix is simply a *rectangular* array of numbers. So any table of data is a matrix.

1. The size of a matrix is indicated by the number of its rows and the number of its columns. A matrix $A$ with $k$ rows and $n$ columns is called a $k \times n$ matrix. The number in row $i$ and column $j$ is called the $(i, j)$ entry and is written as $a_{ij}$.

2. Two matrices are equal if they both have the same size and if the corresponding entries in the two matrices are equal.

3. If a matrix contains only one column (row), it is called a column (row) vector.

Example:

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\] is a $2 \times 2$ matrix.

\[
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
\] is a $2 \times 1$ matrix or a column vector.
(Continued.) **Addition:** One can add two matrices with the same size. E.g.,
\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
a_{11} + 1 & a_{12} \\
a_{12} & a_{22} + 1
\end{bmatrix}.
\]

**Subtraction:** Subtract matrices of the same size simply by subtracting their corresponding entries. E.g.,
\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
- \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
a_{11} - 1 & a_{12} \\
a_{12} & a_{22} - 1
\end{bmatrix}.
\]

**Scalar multiplication:** Matrices can be multiplied by ordinary numbers (scalar). E.g.,
\[
\alpha \begin{bmatrix}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{bmatrix}
= \begin{bmatrix}
\alpha a_{11} & \alpha a_{12} \\
\alpha a_{12} & \alpha a_{22}
\end{bmatrix}.
\]
(Continued.) Matrix multiplication:

- Not all pairs of matrices can be multiplied together, and the order in which matrices are multiplied can matter. We can define the matrix product $AB$ if and only if

  \[
  \text{Number of columns of } A = \text{Number of rows of } B. \quad (32)
  \]

- For the matrix product to exist, $A$ must be $k \times m$ and $B$ must be $m \times n$. To obtain the $(i,j)$ entry of $AB$, multiply the $i$th row of $A$ and the $j$th column of $B$.

- If $A$ is $k \times m$ and $B$ is $m \times n$, then the product $AB$ is $k \times n$. That is, the product matrix $AB$ inherits the number of its rows from $A$ and the number of its columns from $B$.

- E.g.,

  \[
  \begin{bmatrix}
  a_1 & a_2 \\
  \end{bmatrix}
  \begin{bmatrix}
  a_1 \\
  a_2 \\
  \end{bmatrix}
  =
  \begin{bmatrix}
  a_1^2 + a_2^2 \\
  \end{bmatrix}. \quad (33)
  \]

  \[
  \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{12} & a_{22} \\
  \end{bmatrix}
  \begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \end{bmatrix}
  =
  \begin{bmatrix}
  \alpha_1 a_{11} + \alpha_2 a_{12} \\
  \alpha_1 a_{12} + \alpha_2 a_{22} \\
  \end{bmatrix}. \quad (34)
  \]
(Continued.) *Linear dependence* of vectors. A set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) is said to be linearly dependent if and only if any one of them can be expressed as a linear combination of the remaining vectors; otherwise, they are linearly independent.

For any vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) to be linear independent, for any set of scalars \( k_i \), 
\[
\sum_{i=1}^{n} k_i \mathbf{v}_i = 0 \text{ if and only if } k_i = 0 \text{ for all } i.
\]

Example: the three vectors

\[
\mathbf{v}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}
\] (35)

are linearly dependent since \( \mathbf{v}_3 \) is a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \):

\[
3\mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{v}_3
\] (36)
(Continued.) Laws of matrix algebra (think of matrices as generalized numbers)

- **Associative laws:**

  
  $$\begin{align*}
  (A + B) + C &= A + (B + C) \\
  (AB)C &= A(BC)
  \end{align*}$$

  (37)  
  (38)

- **Communication law for addition**

  $$A + B = B + A$$

  (39)

- **Distributive laws**

  $$A(B + C) = AB + AC$$

  (40)

- Note that Communication law for multiplication does NOT hold:

  $$AB \neq BA$$ in general.

  (41)
(Continued.) **Transpose:** the transpose of a $k \times m$ matrix $A$ is the $m \times k$ matrix obtained by interchanging the rows and columns of $A$. It is also written as $A^T$. E.g.,

$$
\begin{bmatrix}
    a_1 & a_2 \\
\end{bmatrix}^T = \begin{bmatrix}
    a_1 \\
    a_2
\end{bmatrix}
$$

(42)

**Square** matrix: a matrix with the same numbers of rows and columns. E.g.,

$$
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
$$

**Determinant** of a square matrix: For $1 \times 1$ matrix $a$, $\det(a) = a$. For $2 \times 2$ matrix $A = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}$,

$$
\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}
$$

(43)

which is just the product of two diagonal entries minus the product of two off-diagonal entries.
(Continued.) *Inverse:* Let $A$ be a square matrix. The matrix $B$ is an inverse for $A$ if $AB = BA = I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called identity matrix (a square matrix with ones in its principal diagonal and zeros everywhere else). If the matrix $B$ exists, we say that $A$ is invertible and denote $B = A^{-1}$.

Squareness is a necessary but not sufficient condition for the existence of an inverse. If a square matrix $A$ has an inverse, $A$ is said to be *nonsingular*. If $A$ has no inverse, it is said to be a singular matrix.
Some properties of the transpose and the inverse

- The operation of matrix transpose and matrix inverse satisfy

\[(AB)' = B'A'\]  \hspace{1cm} (44)
\[(A')' = A\] \hspace{1cm} (45)
\[A'A = 0 \iff A = A' = 0\] \hspace{1cm} (46)
\[(A + B)' = A' + B'\] \hspace{1cm} (47)
\[(A^{-1})' = (A')^{-1}\] \hspace{1cm} (48)
\[(A^{-1})^{-1} = A\] \hspace{1cm} (49)
\[(AB)^{-1} = B^{-1}A^{-1}\] \hspace{1cm} (50)

provided that the objects above are well defined.
(Continued.) System of equations in matrix form: Consider the following two-equation (two unknown variables) system

\begin{align*}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{align*}

(51) (52)

Let \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \) be the coefficient matrix of the system, \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), and \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \). The above equations can be written in a compact form:

\[
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ or } A\mathbf{x} = \mathbf{b}.
\]

(53)

The solution to the above equations can be written as

\[
\mathbf{x} = A^{-1}\mathbf{b},
\]

(54)

if \( A^{-1} \) exists.
The necessary and sufficient conditions for nonsingularity are that the matrix satisfies the \textit{squareness} and \textit{linear independence} conditions (that is, its rows or equivalently, its columns are linearly independent.).

An \(n \times n\) nonsingular matrix \(A\) has \(n\) linearly independent rows (or columns); consequently, it must have rank \(n\). Consequently, an \(n \times n\) matrix having rank \(n\) must be nonsingular.

The concept of \textit{determinant} can be used to not only determine whether a square matrix is nonsingular, but also calculate the inverse of the matrix.
(Continued.) The value of a determinant of order \( n \) can be found by the Laplace expansion of any row or any column as follows

\[
|A| = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} |M_{ij}|, \tag{55}
\]

where \( |M_{ij}| \) is the *minor* of the element \( a_{ij} \) and can be obtained by deleting the \( i \)th row and \( j \)th column of the determinant \( |A| \) and \( |C_{ij}| = (-1)^{i+j} |M_{ij}| \) is called *cofactor*.

Example: for a \( 3 \times 3 \) matrix \( A \), its determinant has the value:

\[
|A| = \begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11} \begin{vmatrix}
  a_{22} & a_{23} \\
  a_{32} & a_{33}
\end{vmatrix} - a_{12} \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix} + a_{13} \begin{vmatrix}
  a_{21} & a_{22} \\
  a_{31} & a_{32}
\end{vmatrix}. \tag{56}
\]

\[
= a_{11} \begin{vmatrix}
  a_{22} & a_{23} \\
  a_{32} & a_{33}
\end{vmatrix} - a_{12} \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix} + a_{13} \begin{vmatrix}
  a_{21} & a_{22} \\
  a_{31} & a_{32}
\end{vmatrix}. \tag{57}
\]
Given a linear equation system $Ax = b$, where $A$ is a square coefficient matrix, we have

$$|A| \neq 0 \iff A \text{ is row or column independent}$$

$$\iff A \text{ is nonsingular}$$

$$\iff A^{-1} \text{ exists}$$

$$\iff A \text{ unique solution } x = A^{-1}b \text{ exists.}$$
Finding the inverse matrix

Each element \( a_{ij} \) of \( n \times n \) matrix \( A \) has a cofactor \( |C_{ij}| \). A cofactor matrix \( C = [|C_{ij}|] \) is also \( n \times n \). The transpose \( C' \) is referred to as the adjoint of \( A \) and denoted by \( \text{adj} \ A \).

The procedure to calculate \( A^{-1} \):

1. Find \( |A| \)
2. Find the cofactors of all elements of \( A \) and form \( C = [|C_{ij}|] \)
3. Form \( C' = \text{adj} \ A \)
4. Determine

\[
A^{-1} = \frac{\text{adj} \ A}{|A|} \quad (58)
\]
Another way to calculate the inverse of a matrix $A_{n \times n}$ is to use the Cramer rule. Let $B_j$ be the $n \times n$ matrix formed by taking $A$ and substituting its $j$–th column, $a_j$, with the constants vector $b$. For example, for $j = 2$,

$$
B_2 = \begin{bmatrix}
a_1 & b & a_3 & \cdots & a_n \\
a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\
a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
a_{n1} & b_n & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
$$

Let $|A| \neq 0$ and $|B_j|$ be the determinants of $A$ and $B_j$, respectively. See any linear algebra textbook or pages 103-104 in CW for proof.

Cramer rule then says that the $j$–th element of the solution $\mathbf{x} = A^{-1} \mathbf{b}$ is given by

$$
x_j = \frac{|B_j|}{|A|} \quad \forall j = 1, \ldots, n.
$$
Reconsider the two-commodity Model

\[ c_1 P_1 + c_2 P_2 = -c_0 \]  \hspace{1cm} (62)
\[ d_1 P_1 + d_2 P_2 = -d_0, \]  \hspace{1cm} (63)

which can be rewritten as

\[
\begin{bmatrix}
  c_1 & c_2 \\
  d_1 & d_2 \\
\end{bmatrix}
\begin{bmatrix}
  P_1 \\
  P_2 \\
\end{bmatrix}
= \begin{bmatrix}
  -c_0 \\
  -d_0 \\
\end{bmatrix},
\]  \hspace{1cm} (64)

which implies that

\[
\begin{bmatrix}
  P_1 \\
  P_2 \\
\end{bmatrix}
= \begin{bmatrix}
  c_1 & c_2 \\
  d_1 & d_2 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
  -c_0 \\
  -d_0 \\
\end{bmatrix}
= \frac{1}{|A|}
\begin{bmatrix}
  d_2 & -c_2 \\
  -d_1 & c_1 \\
\end{bmatrix}
\begin{bmatrix}
  -c_0 \\
  -d_0 \\
\end{bmatrix}
= \begin{bmatrix}
  \frac{c_2 d_0 - c_0 d_2}{c_1 d_2 - c_2 d_1} \\
  \frac{c_1 d_2 - c_2 d_1}{c_0 d_1 - c_1 d_0} \\
  \frac{c_0 d_1 - c_1 d_0}{c_1 d_2 - c_2 d_1} \\
\end{bmatrix}.
\]  \hspace{1cm} (65)
Reconsider equilibrium national-income Model

- The model

\[
Y = C + l_0 + G_0 \text{ (Equilibrium condition)} \tag{66}
\]

\[
C = a + bY \text{ (The consumption function)} \tag{67}
\]

can be rewritten in a compact matrix form

\[
\begin{bmatrix}
1 & -1 \\
-b & 1 \\
\end{bmatrix}
\begin{bmatrix}
Y \\
C \\
\end{bmatrix}
= \begin{bmatrix}
l_0 + G_0 \\
a \\
\end{bmatrix},
\tag{68}
\]

which implies that

\[
\begin{bmatrix}
Y \\
C \\
\end{bmatrix}
= \frac{1}{1-b}
\begin{bmatrix}
1 & 1 \\
b & 1 \\
\end{bmatrix}
\begin{bmatrix}
l_0 + G_0 \\
a \\
\end{bmatrix}
= \begin{bmatrix}
a + l_0 + G_0 \\
\frac{a + b(l_0 + G_0)}{1-b} \\
\end{bmatrix} \tag{69}
\]

- Applying Cramer rule gives:

\[
Y = \left| \begin{bmatrix}
l_0 + G_0 & -1 \\
a & 1 \\
\end{bmatrix} \right| / \left| \begin{bmatrix}
1 & -1 \\
-b & 1 \\
\end{bmatrix} \right| = \frac{a + l_0 + G_0}{1-b}. \tag{70}
\]
Consider another linear model of macroeconomy: the IS-LM model made up of two sectors: the real goods sector and the monetary sector. The goods market involves the following equations:

\[ Y = C + I + G \]  
\[ C = a + b(1 - t)Y \]  
\[ I = d - ei \]  
\[ G = G_0, \]

where \( Y, C, I, \) and \( i \) (the interest rate) are endogenous variables, \( G_0 \) is the exogenous variable, and \( a, b, d, e, \) and \( t \) are structural parameters.

In the newly introduced money market, we have

\[ M_d = M_s \] (Equilibrium condition in the money market)  
\[ M_d = kY - li \] (Aggregate money demand)  
\[ M_s = M_0, \]

where \( M_0 \) is the exogenous money supply and \( k \) and \( l \) are parameters. Note that these three equations can be reduced to

\[ M_0 = kY - li \]
Together, the two sectors give the following system of equations

\[
Y - C - I = G_0 \tag{79}
\]

\[
b(1 - t)Y - C = -a \tag{80}
\]

\[
l + ei = d \tag{81}
\]

\[
kY - li = M_0 \tag{82}
\]

which can be written in a compact matrix form:

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
-b(1 - t) & -1 & 0 & 0 \\
0 & 0 & 1 & e \\
k & 0 & 0 & -l
\end{bmatrix}
\begin{bmatrix}
Y \\
C \\
l \\
i
\end{bmatrix}
= 
\begin{bmatrix}
G_0 \\
-a \\
d \\
M_0
\end{bmatrix}
\]
(Continued.) Applying Cramer rule gives:

\[
Y^* = \begin{vmatrix}
G_0 & -1 & -1 & 0 \\
-a & -1 & 0 & 0 \\
d & 0 & 1 & e \\
M_0 & 0 & 0 & -l \\
\end{vmatrix}
/ \begin{vmatrix}
1 & -1 & -1 & 0 \\
b(1-t) & -1 & 0 & 0 \\
0 & 0 & 1 & e \\
k & 0 & 0 & -l \\
\end{vmatrix}
= \frac{l(a + d + G_0) + eM_0}{ek + l[1 - b(1 - t)]}.
\]

(83)

Similarly, we may derive the expressions for other endogenous variables \((C, l, i)\) in terms of both exogenous variables and structural parameters.

It is clear that Cramer's rule is elegant, but if you have to invert a numerical matrix (e.g., \(a = 1, t = 0.1, b = 0.4, \) etc.), you’d better to use Matlab, Gauss, Mathematica, etc.
The equation system $Ax = b$ considered before can have any constants in the vector $b$. If $b = 0$, that is, every elements in the $b$ vector is zero, the equation system becomes

$$Ax = 0,$$

where $0$ is a zero vector. This special case is referred to as a homogeneous-equation system. “homogeneous” here means that if all the variables $x_1, \ldots, x_n$ are multiplied by the same number, the equation system will remain valid. This is possible only if $b = 0$.

Note that if the coefficient matrix $A$ is nonsingular (that is, it is invertible), a homogeneous-equation system can yield only a trivial solution, that is, $x = 0$. (check it yourself)
(Continued.) The only way to get a nontrivial solution from the above HE system is to have

$$|A| = 0,$$

that is, to have a singular coefficient matrix $A$.  

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No unique nontrivial solution exists for this HE system. For example, consider a $2 \times 2$ case:

$$a_{11} x_1 + a_{12} x_2 = 0$$  
$$a_{21} x_1 + a_{22} x_2 = 0,$$

assume that $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is singular, so that $|A| = 0$. This implies that the row vector $\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ is a multiple of the row vector $\begin{bmatrix} a_{21} & a_{22} \end{bmatrix}$; consequently, one of the two equations is redundant. By deleting the second equation from the system, we end up with one equation in two variables and the solution is

$$x_1 = \left(-\frac{a_{12}}{a_{11}} \right) x_2,$$

which is well-defined if $a_{11} \neq 0$ and represents an infinite number of solutions.
### Solution Outcomes for a Linear-equation System

$$Ax = b$$

**Vector $b$**

| $b \neq 0$ | There exists a unique, nontrivial solution $x^* \neq 0$ |
| $|A| \neq 0$ | Infinite numbers of solutions if equations are dependent |
| $|A| = 0$ | No solution if equations are inconsistent |

| $b = 0$ | There exists a unique, trivial solution $x^* = 0$ |
| $|A| \neq 0$ | Infinite numbers of solutions if equations are dependent |
| $|A| = 0$ | Not possible if equations are inconsistent |