Lecture 6: Discrete-Time Dynamic Optimization

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In static optimization, the task is to find a single value for each control variable, such that the objective function will be maximized or minimized. In contrast, in a dynamic setting, time enters explicitly and we encounter a dynamic optimization problem. In such a problem, we need to find the optimal time path of control and state during an entire planning period.

Specifically, at any time $t$, we have to choose the value of some control variable, $c(t)$, which will then affect the value of the state variable, $x(t + 1)$, via a state transition equation given the current state $x(t)$.

Two leading methods to solve DO problems are: optimal control (OC) and dynamic programming (DP). They can be applied in deterministic or stochastic and discrete-time or continuous-time settings.
Since our objective is to maximize the profit over the entire period, the objective should take the form of summation from $t = 0$ to $t = T$ in the discrete-time setting. The problem also specifies the initial value of the state variable $x$, $x(0)$, and the terminal condition of $y$, $x(T + 1)$:

$$\max_{\{c(t), x(t+1)\}} \sum_{t=0}^{T} r(c(t), x(t), t),$$

subject to

$$x(t + 1) - x(t) = f(c(t), x(t), t),$$

$$x_0 = x(0), x(T + 1) \geq 0,$$

for any $t \geq 0$. 
(Conti.) As in the static optimization case, the Lagrangian for this intertemporal (dynamic) problem is

\[ L = \sum_{t=0}^{T} \{ r(c_t, x_t, t) + \lambda_t [f(c_t, x_t, t) + x_t - x_{t+1}] \} + \lambda_{T+1} x_{T+1}, \]

where \( \lambda_t \) is the shadow price (i.e., the Lagrange multiplier).

The FOCs for \( c_t \) are:

\[ \frac{\partial L}{\partial c_t} = r'_c(c_t, x_t, t) + \lambda_t f'_c(c_t, x_t, t) = 0 \text{ for } t = 0, 1, \ldots, T. \]

The FOCs for \( x_{t+1} \) are complicated because each \( x_t \) appears in two terms in the Lagrangian function.
(Conti.) Consider some relevant terms in the above Lagrangian function:

\[ \sum_{t=0}^{T} \lambda_t (x_t - x_{t+1}) = (\lambda_0 x_0 - \lambda_T x_{T+1}) + \sum_{t=0}^{T-1} x_{t+1} (\lambda_{t+1} - \lambda_t). \]

The FOCs for \( x_{t+1} \):

\[
\frac{\partial L}{\partial x_{t+1}} = r'_x(c_{t+1}, x_{t+1}, t + 1) + \lambda_{t+1} f'_x(c_{t+1}, x_{t+1}, t + 1) + \lambda_{t+1} - \lambda_t = 0,
\]

for \( t = 0, \ldots, T - 1 \).
Theorem (The Maximum Principle)

The necessary FOCs for maximizing (1) subject to (2) are:
(i) \( \frac{\partial L_t}{\partial c_t} = 0 \) for all \( t = 0, \ldots, T \).
(ii) \( x_{t+1} - x_t = f(c_t, x_t, t) \) for all \( t \) (The state transition equation)
(iii) \( r'_x(c_{t+1}, x_{t+1}, t+1) + \lambda_{t+1} f'_x(c_{t+1}, x_{t+1}, t+1) + \lambda_{t+1} - \lambda_t = 0 \) for all \( t \) (The costate \( (\lambda_t) \) equation)
(iv) \( x_{T+1} \geq 0, \lambda_{T+1} \geq 0, x_{T+1} \lambda_{T+1} = 0 \) (The transversality condition, or the complementary-slackness condition)
Example: Optimal Consumption-Saving Model

- Suppose that a consumer has the utility function \( u(c_t) \), where \( c_t \) is consumption at time \( t \). The consumer’s utility is concave:

\[
    u' > 0, \quad u'' < 0, \tag{6}
\]

and \( u(c_t) \) satisfies the Inada conditions (They ensure that consumption will always be *interior*):

\[
    \lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0. \tag{7}
\]

- The consumer is also endowed with \( a_0 = a(0) \), and has income stream derived from holding the asset: \( y_t = r a_t \), where \( r \) is the interest rate. The consumer uses the income to purchase \( c \). Any income not consumed is added to the asset holdings as savings:

\[
    a_{t+1} = (1 + r) a_t - c_t. \tag{8}
\]
The consumer’s lifetime utility maximization problem is to

\[
\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{T} \beta^t u(c_t)
\]

subject to

\[
a_{t+1} = Ra_t - c_t, \quad t = 0, \ldots, T,
\]

\[
a_0 = a(0), \quad a_{T+1} \geq 0,
\]

where \(\beta\) is the consumer’s rate of time preference \((\beta \leq 1)\) and \(R = 1 + r\) is the gross interest rate.
(Conti.) Since \( \lim_{c \to 0} u'(c) = \infty \), \( c(t) > 0 \) for all \( t \leq T \). Further, to ensure that a solution exists, assets are constrained to be nonnegative after the terminal period \( T \) (i.e., when the consumer dies), \( a_{T+1} \geq 0 \).

Note that this nonnegativity constraint must bind, i.e., \( a_{T+1} = 0 \). In fact, this constraint serves two important purposes:

- First, without this constraint \( a_{T+1} \geq 0 \), the consumer would want to set \( a_{T+1} = -\infty \) and die with outstanding debts. This is clearly not feasible.
- Second, the fact that the constraint is binding in the optimal solution guarantees that resources are not being thrown away after \( T \).
Using Optimal Control to Solve the Model

- Setting up the Lagrangian:

\[
L = \sum_{t=0}^{T} \beta^t \left\{ u(c_t) + \lambda_t [Ra_t - c_t - a_{t+1}] \right\} + \lambda_{T+1} a_{T+1} \quad (12)
\]

- The FOC for \( c_t \) is \( \frac{\partial L}{\partial c_t} = 0 \):

\[
u'(c_t) - \lambda_t = 0 \text{ for } t = 0, 1, \ldots, T. \quad (13)\]

- The FOC for \( a_{t+1} \) is \( \frac{\partial L}{\partial a_{t+1}} = 0 \):

\[-\lambda_t + \beta R \lambda_{t+1} = 0 \text{ for } t = 0, \ldots, T - 1. \quad (14)\]
The terminal condition $a_{T+1} \geq 0$ implies that the transversality condition (the complementary slackness condition) is

$$\lambda_{T+1} \geq 0, a_{T+1} \geq 0, a_{T+1} \lambda_{T+1} = 0,$$

(15)

which means that either the asset holdings ($a$) must be exhausted on the terminal date, or the shadow price of capital ($\lambda_t$) must be 0 on the terminal date. Since $u' > 0$, the marginal value of capital ($\lambda$) cannot be 0 and thus the capital stock should optimally be exhausted by the terminal date $T + 1$, i.e., $a_{T+1} = 0$. 
(Conti.) Combining (13) with (14) gives

\[ u'(c_t) = \beta Ru'(c_{t+1}), \]  

(16)

which is called the Euler equation characterizing optimal consumption dynamics.

Suppose that the utility function has an isoelastic form

\[ u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \]  

(17)

where \( \gamma \geq 0 \) (Note that when \( \gamma = 1 \), the function reduces to \( \ln c_t \)). We have

\[ c_t^{-\gamma} = \beta Rc_t^{-\gamma} \implies \frac{c_{t+1}}{c_t} = (\beta R)^{1/\gamma}. \]  

(18)

Therefore, if \( \beta R > 1 \), the optimal consumption will rise over time; if \( \beta R < 1 \), the optimal consumption will decline over time. When \( \beta R = 1 \), \( c_t = c_{t+1} \).
(Conti.) (18) implies that the elasticity of intertemporal substitution (EIS) is

\[
\frac{d \left( \frac{c_{t+1}}{c_t} \right)}{dR / R} = \frac{d \ln \left( \frac{c_{t+1}}{c_t} \right)}{d \ln R} = \frac{1}{\gamma}.
\]

Hence, increasing \( \gamma \) (i.e., reducing \( \frac{1}{\gamma} \)) make the agent more unwilling to postpone consumption (i.e., more unwilling to save). That’s why we call this type of utility functions the isoelastic utility function.

The optimal problem is pinned down by a given initial condition \( (a_0) \) and by a terminal condition \( (a_{T+1} = 0) \). The sets of \( (T - 1) \) Euler equations (18) and constraints (10) then determine the time path of \( c \) as well as \( a \).
The original budget constraint (10) implies that

\begin{align*}
    a_1 &= R a_0 - c_0, \\
    a_2 &= R a_1 - c_1, \\
    \cdots \\
    a_{T+1} &= R a_T - c_T.
\end{align*}

\begin{equation}
    \frac{a_{T+1}}{R^T} + \left( \frac{c_T}{R^T} + \cdots + \frac{c_1}{R} + c_0 \right) = R a_0 \quad \Rightarrow \\
    \sum_{t=0}^{T} \frac{c_t}{R^t} = R a_0,
\end{equation}

where we use the fact that $a_{T+1} = 0$. 
(Conti.) For simplicity, assume that $\gamma = 1$. Combining (18) with (20) gives

$$c_0 + \frac{1}{R} (\beta Rc_0) + \cdots + \frac{1}{R^T} \left( \beta^T R^T c_0 \right) = Ra_0 \Rightarrow
$$

$$c_0 = \frac{R}{1 + \beta + \cdots + \beta^T a_0}. \quad (21)$$

Hence, the time path of optimal consumption is

$$c_t = \frac{R^{t+1} \beta^t}{1 + \beta + \cdots + \beta^T a_0} \quad (22)$$

where $t = 0, \cdots, T$. Given $c_t$, it is straightforward to determine the time path of $a_t$:

$$a_{t+1} = Ra_t - c_t = Ra_t - \frac{R^{t+1} \beta^t}{1 + \beta + \cdots + \beta^T a_0}, \quad (23)$$

where $t = 0, \cdots, T$ and $a_{T+1} = 0$. 
(Conti.) Consider an important special case, \( \beta R = 1 \). In this case, (18) implies that \( c_0 = c_1 = \cdots = c_T \),

\[
c_0 \left( 1 + \cdots + \frac{1}{R^T} \right) = Ra_0 \quad \implies \quad c_t = \frac{R - 1}{1 - R^{-(T+1)}} a_0.
\]  

(24)

where \( t = 0, \cdots, T \).

Next, the optimal path of \( a_t \) can be derived by solving the following difference equation:

\[
a_{t+1} = Ra_t - c_t = Ra_t - \frac{r}{1 - R^{-(T+1)}} a_0 \quad \implies \quad a_t = \left( 1 - \frac{1}{1 - R^{-(T+1)}} \right) a_0 R^t + \frac{1}{1 - R^{-(T+1)}} a_0,
\]  

(25)
Instead of defining the Lagrange multiplier for each flow budget constraint, we may define a Lagrange multiplier for the intertemporal budget constraint (that is, the lifetime budget constraint):

$$\sum_{t=0}^{T} \frac{c_t}{R_t} = Ra_0,$$

where we have used the fact that $a_{T+1} = 0$. That is, we may write the Lagrangian as follows

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left( Ra_0 - \sum_{t=0}^{T} \frac{c_t}{R_t} \right),$$

where $\lambda$ is the constant Lagrange multiplier for the lifetime budget constraint.
(Conti.) The FOCs for an optimum are then

\[ \beta^t u'(c_t) = \lambda \frac{1}{R^t}, \]  
where \( t = 0, \ldots, T. \)  

(26)

Since \( \lambda \) is a constant, the above FOCs implies that the Euler equations are

\[ u'(c_t) = \beta Ru'(c_{t+1}), \]  
where \( t = 0, \ldots, T - 1, \)

which is identical to those Euler equations we derived above.
There are some reasons to consider the infinite horizon ($T = \infty$) case:

- People don’t live forever, but they may care about their offspring (a bequest motive).
- Assuming an infinite horizon eliminates the terminal date complications (stop saving, consume their entire value, etc. There is no real world counterpart for these actions because the real economy continue forever).
- Many economic models with a long time horizon tend to show very similar results to IH models if the horizon is long enough.
- IH models are *stationary in nature*: the remaining time horizon doesn't change as we move forward in time.
The consumer’s lifetime utility maximization problem

\[
\max_{c(t)} \sum_{t=0}^{\infty} \beta^t u(c_t)
\]  

subject to \(a_{t+1} = Ra_t - c_t, \forall t \geq 0\), given the initial asset holdings \(a_0 = a(0)\).

In addition, we need to impose the following institutional assumption: 
**no Ponzi game condition (nPg):**

\[
\lim_{t \to \infty} \left( \frac{a_{t+1}}{R_t} \right) \geq 0,
\]

means that in present value terms, the agent cannot engage in borrowing and lending so that his terminal asset holdings are negative.
(Conti.) (36) can help obtain an *intertemporal* budget constraint:

\[
\sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{a_{T+1}}{R^T} = Ra_0.
\]

Applying (36), we have

\[
\lim_{T \to \infty} \left( \sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{a_{T+1}}{R^T} \right) = \lim_{T \to \infty} \sum_{t=0}^{T} \frac{c_t}{R^t} + \lim_{T \to \infty} \left( \frac{a_{T+1}}{R^T} \right)
\]

\[
= \sum_{t=0}^{\infty} \frac{c_t}{R^t} + \lim_{T \to \infty} \left( \frac{a_{T+1}}{R^T} \right) \implies \sum_{t=0}^{\infty} \frac{c_t}{R^t} \leq Ra_0.
\]

(29)
(Conti.) In an optimum, (37) must be binding:

$$\sum_{t=0}^{\infty} \frac{c_t}{R^t} = Ra_0 \tag{30a}$$

because (36) is binding, i.e., \( \lim_{T \to \infty} \left( \frac{a_t+1}{R^t} \right) = 0 \) (Otherwise, the agent can consume the left amount and reach a higher utility level, which contradicts optimum).

Set up the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u \left( c_t \right) + \lambda \left( Ra_0 - \sum_{t=0}^{\infty} \frac{c_t}{R^t} \right)$$

The FOCs w.r.t. \( c_t \) for any \( t \) are

$$\beta^t u' \left( c_t \right) - \lambda \frac{1}{R^t} = 0, \ \forall t \geq 0, \tag{31}$$

where \( \lambda \) is the Lagrangian multiplier associated with the intertemporal budget constraint.
(Conti.) Note that the FOCs imply the following Euler equation linking consumption in two consecutive periods

\[ u'(c_t) = \beta Ru'(c_{t+1}), \forall t \geq 0. \]  

(32)

Suppose that the utility function is log, \( u(c_t) = \log c_t \). (39) implies that \( c_t = (\beta R)^t c_0, \forall t \geq 1 \). Substituting them into the (38a) gives

\[
\sum_{t=0}^{\infty} \frac{(\beta R)^t}{R^t} c_0 = Ra_0 \implies \left( \sum_{t=0}^{\infty} \beta^t \right) c_0 = Ra_0 \implies c_0 = R (1 - \beta) a_0,
\]

and consumption in periods \( t \geq 1 \) can be recovered:

\[ c_t = (\beta R)^t c_0 = (\beta R)^t R (1 - \beta) a_0, \forall t \geq 1. \]  

(34)

Similarly, we can determine the optimal time path of asset holdings:

\[ a_{t+1} = Ra_t - (\beta R)^t R (1 - \beta) a_0. \]
The Infinite Horizon Case

- There are some reasons to consider the infinite horizon \((T = \infty)\) case:
  - People don’t live forever, but they may care about their offspring (a bequest motive).
  - Assuming an infinite horizon eliminates the terminal date complications (stop saving, consume their entire value, etc. There is no real world counterpart for these actions because the real economy continue forever).
  - Many economic models with a long time horizon tend to show very similar results to IH models if the horizon is long enough.
  - IH models are _stationary in nature_: the remaining time horizon doesn’t change as we move forward in time.
The consumer’s lifetime utility maximization problem

\[
\max_{c(t)} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{35}
\]

subject to \(a_{t+1} = Ra_t - c_t, \forall t \geq 0\), given the initial asset holdings \(a_0 = a(0)\).

In addition, we need to impose the following institutional assumption: no Ponzi game condition (nPg):

\[
\lim_{t \to \infty} \left( \frac{a_{t+1}}{R^t} \right) \geq 0, \tag{36}
\]

means that in present value terms, the agent cannot engage in borrowing and lending so that his terminal asset holdings are negative.
(Conti.) (36) can help obtain an intertemporal budget constraint:

\[ \sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{a_{T+1}}{R^T} = Ra_0. \]

Applying (36), we have

\[
\lim_{T \to \infty} \left( \sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{a_{T+1}}{R^T} \right) = \lim_{T \to \infty} \sum_{t=0}^{T} \frac{c_t}{R^t} + \lim_{T \to \infty} \left( \frac{a_{T+1}}{R^T} \right)
\]

\[
= \sum_{t=0}^{\infty} \frac{c_t}{R^t} + \lim_{T \to \infty} \left( \frac{a_{T+1}}{R^T} \right) \implies \\
\sum_{t=0}^{\infty} \frac{c_t}{R^t} \leq Ra_0. \tag{37}
\]
(Conti.) In an optimum, (37) must be binding:

$$\sum_{t=0}^{\infty} \frac{c_t}{R_t} = R a_0 \quad (38a)$$

because (36) is binding, i.e., \( \lim_{T \to \infty} \left( \frac{a_t+1}{R_t} \right) = 0 \) (Otherwise, the agent can consume the left amount and reach a higher utility level, which contradicts optimum).

Set up the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left( R a_0 - \sum_{t=0}^{\infty} \frac{c_t}{R_t} \right)$$

The FOCs w.r.t. \( c_t \) for any \( t \) are

$$\beta^t u'(c_t) - \lambda \frac{1}{R_t} = 0, \forall t \geq 0, \quad (39)$$

where \( \lambda \) is the Lagrangian multiplier associated with the intertemporal budget constraint.
(Conti.) Note that the FOCs imply the following Euler equation linking consumption in two consecutive periods

\[ u'(c_t) = \beta R u'(c_{t+1}), \forall t \geq 0. \] (40)

Suppose that the utility function is log, \( u(c_t) = \log c_t \). (39) implies that \( c_t = (\beta R)^t c_0, \forall t \geq 1 \). Substituting them into the (38a) gives

\[
\sum_{t=0}^{\infty} \frac{(\beta R)^t}{R^t} c_0 = Ra_0 \implies \left( \sum_{t=0}^{\infty} \beta^t \right) c_0 = Ra_0 \implies \\
c_0 = R (1 - \beta) a_0, \] \] (41)

and consumption in periods \( t \geq 1 \) can be recovered:

\[ c_t = (\beta R)^t c_0 = (\beta R)^t R (1 - \beta) a_0, \forall t \geq 1. \] (42)

Similarly, we can determine the optimal time path of asset holdings:

\[ a_{t+1} = Ra_t - (\beta R)^t R (1 - \beta) a_0. \]
The key difference bw the RCK model and the Solow model is that the saving rate is endogenized in the RCK model. We posit a single representative agent in the economy and has the following preference

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$  \hspace{1cm} (43)

where $u(c_t)$ also satisfies the same conditions as before.

Abstracting from population growth or technological progress, the resource constraint facing the representative consumer at any $t$ are

$$k_{t+1} = (1 - \delta) k_t + f(k_t) - c_t,$$  \hspace{1cm} (44)

where $k_0$ is given. $\delta \in (0, 1)$ is the depreciation rate, the production function $f(k_t)$ satisfies the usual conditions: $f'(k_t) > 0$ and $f''(k_t) < 0$. 
Solving the Model

- As usual, we first set up the Lagrangian:

\[ L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[ (1 - \delta) k_t + f(k_t) - c_t - k_{t+1} \right] \right\} \]  

(45)

where \( \lambda_t \geq 0 \) denote the multiplier on the resource constraint at time \( t \).

- The FOCs w.r.t. \( c_t \) and \( k_{t+1} \) are

\[ u'(c_t) = \lambda_t \]  

(46)

\[ \beta \left[ 1 - \delta + f'(k_{t+1}) \right] \lambda_{t+1} = \lambda_t \]  

(47)

- Combining both FOCs can generate the Euler equation

\[ \beta \left[ 1 - \delta + f'(k_{t+1}) \right] u'(c_{t+1}) = u'(c_t) \]  

(48)
(conti.) We also need the transversality condition (TVC) to guarantee that asymptotically the shadow value of more capital is zero:

$$ \lim_{t \to \infty} \beta^t \lambda_t k_{t+1} = \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0, $$

(49)

where $\beta^t \lambda_t$ is the present-value utility evaluation of an additional unit of resources in period $t$.

Hence, the TVC says that the value (discounted into PV utilities) of each additional unit of capital at infinity times the actual amount of capital has to be zero. Otherwise, the consumer can modify such a capital path and increase consumption for an overall increase in utility without violating feasibility. In the finite horizon case, the corresponding condition is that $k_{T+1} = 0$. 
(conti.) The nPg condition discussed in the last lecture and the TVC here play a very similar roles in dynamic optimization in a purely mechanical sense. In fact, they are the same condition when the FOCs are satisfied:

\[
\lim_{t \to \infty} \beta^t \lambda_t k_{t+1} = 0 \iff \lim_{t \to \infty} \left[ (R_1^{-1}) (R_2^{-1}) \cdots (R_t^{-1}) \right] \lambda_0 k_{t+1} = \lambda_0 \lim_{t \to \infty} R_t k_{t+1} = 0,
\]

where \( R_{t+1} = 1 - \delta + f'(k_{t+1}) \) and \( \lambda_0 = u'(c_0) \) is finite.

However, the two conditions are conceptually very different: The nPg is a restriction on the choices of the agent. In contrast, the TVC is a prescription how to behave optimally, given a choice set.
We have formed a nonlinear difference equation system in terms of $c_t$ and $k_t$ that can characterize the economy completely:

$$u'(c_t) = \beta [1 - \delta + f'(k_{t+1})] u'(c_{t+1})$$  \hspace{1cm} (50)

$$k_{t+1} = (1 - \delta) k_t + f(k_t) - c_t$$

with two boundary conditions: the initial condition $k_0 = k(0)$ and the transversality condition (TVC) (49).

Now we can determine the intertemporal equilibrium (the steady state) of the above dynamic system, i.e., $k_{t+1} = k_t = \bar{k}$ and $c_{t+1} = c_t = \bar{c}$:

$$1 = \beta [1 - \delta + f'(\bar{k})]$$  \hspace{1cm} (51)

$$0 = -\delta \bar{k} + f(\bar{k}) - \bar{c}$$  \hspace{1cm} (52)
(conti.) Note that the first equation is independent of $\bar{c}$ and can easily determine $\bar{k}$:

$$f'(\bar{k}) = \frac{1}{\beta} - 1 + \delta = \rho + \delta,$$

where $\rho = \frac{1}{\beta} - 1$ is called the time discount rate. Note that if $f(k_t) = A k_t^\alpha$,

$$\bar{k} = \left( \frac{A \alpha}{\rho + \delta} \right)^{1/(1-\alpha)}.$$

When $\bar{k}$ is determined,

$$\bar{c} = f(\bar{k}) - \delta \bar{k}.$$
For this nonlinear system, we can use *the phase diagram* to analyze its qualitative dynamics. More specifically, assume that the utility function is $\ln c_t$, and rewrite the dynamic system as:

$$\frac{c_{t+1}}{c_t} = \beta (1 - \delta + f'(k_{t+1})),$$

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t,$$

which means

$$d \left( \frac{c_{t+1}}{c_t} \right)_{dk_{t+1}} = \beta f''(k_{t+1}) < 0 \quad \text{and} \quad \frac{d (k_{t+1} - k_t)}{dc_t} = -1 < 0$$

and from which we can have two demarcation lines separating the space. It turns out the intertemporal equilibrium is saddle-point equilibrium. [Insert Figure here]
Alternatively, we can use the linearization method to approximate the original nonlinear system around the intertemporal equilibrium and then solve the resulting linear difference equation system.

Linearizing the above dynamic system around \((\bar{k}, \bar{c})\) gives

\[
\begin{align*}
c_{t+1} &= \bar{c} + \beta \left( 1 - \delta + A\alpha \bar{k}^{\alpha - 1} \right) (c_t - \bar{c}) + (\alpha - 1) \beta \bar{c} A\alpha \bar{k}^{\alpha - 2} (k_{t+1} - \bar{k}) + c_t \\
k_{t+1} &= \bar{k} - (c_t - \bar{c}) + \left( 1 - \delta + A\alpha \bar{k}^{\alpha - 1} \right) (k_t - \bar{k}) ,
\end{align*}
\]

where \(\frac{1}{\beta} = 1 - \delta + A\alpha \bar{k}^{\alpha - 1}\) and which can be written (denote \(\tilde{x}_{t+1} = x_{t+1} - \bar{x}, x = k, c\))

\[
\begin{bmatrix}
1 & - (\alpha - 1) \beta \bar{c} A\alpha \bar{k}^{\alpha - 2} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{c}_{t+1} \\
\tilde{k}_{t+1}
\end{bmatrix}
+ \begin{bmatrix}
-1 & 0 \\
1 & -\frac{1}{\beta}
\end{bmatrix}
\begin{bmatrix}
\tilde{c}_t \\
\tilde{k}_t
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\(57\)
Multiplying $J^{-1}$ on both sides gives

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{c}_{t+1} \\
\tilde{k}_{t+1} \\
\end{bmatrix}
+ J^{-1}
\begin{bmatrix}
-1 & 0 \\
1 & -\frac{1}{\beta} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{c}_t \\
\tilde{k}_t \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
$$

where we have substituted $\bar{k} = \left(\frac{A\alpha}{\rho + \delta}\right)^{1/(1-\alpha)}$ into the matrix $M$ and

$$
K = J^{-1}M = \begin{bmatrix}
1 & (\alpha - 1) \beta \bar{c} A\alpha \bar{k}^{-\alpha - 2} \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
1 & -\frac{1}{\beta} \\
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-1 + (\alpha - 1) \beta \bar{c} A\alpha \bar{k}^{-\alpha - 2} - (\alpha - 1) \bar{c} A\alpha \bar{k}^{-\alpha - 2} \\
1 & -\frac{1}{\beta} \\
\end{bmatrix}.
$$
The characteristic roots $b$ can be obtained by solving the equation:

$$|bl + K| = 0 \implies \begin{vmatrix} b - 1 + (\alpha - 1) \beta \bar{c} A \alpha \bar{k}^{\alpha - 2} & - (\alpha - 1) \bar{c} A \alpha \bar{k}^{\alpha - 2} \\ 1 & b - \frac{1}{\beta} \end{vmatrix} = 0$$

$$b^2 - \left[ 1 + (1 - \alpha) \beta \bar{c} A \alpha \bar{k}^{\alpha - 2} + (1 + \rho) \right] b + \left[ 1 + (1 - \alpha) \beta \bar{c} A \alpha \bar{k}^{\alpha - 2} \right] (1 + \rho) + (\alpha - 1) \bar{c} A \alpha \bar{k}^{\alpha - 2} = 0 \implies$$

$$\text{trace} = b_1 + b_2 = 1 + (1 - \alpha) \beta \bar{c} A \alpha \bar{k}^{\alpha - 2} + \frac{1}{\beta} > 2,$$

$$\det = b_1 b_2 = \left[ 1 + (1 - \alpha) \beta \bar{c} A \alpha \bar{k}^{\alpha - 2} \right] \frac{1}{\beta} + (\alpha - 1) \bar{c} A \alpha \bar{k}^{\alpha - 2}$$

$$= \frac{1}{\beta} > 1$$
(conti.) Hence, the discriminant should be positive because

\[ \Delta = \text{trace} (J_E)^2 - 4 \det (J_E) = \left[ 1 + (1 - \alpha) \beta \bar{c} A \alpha k^{\alpha - 2} + \frac{1}{\beta} \right]^2 - 4 \frac{1}{\beta} \]

\[ = \left[ 1 + (1 - \alpha) \beta \bar{c} A \alpha k^{\alpha - 2} - \frac{1}{\beta} \right]^2 > 0 \]

which means that both roots are real. Also, because \( \det = \frac{1}{\beta} > 1 \) and \( \text{trace} > 2 \), the two roots must individually be positive. We can also judge the magnitudes of the two roots as follows:

\[ |b1 + K| = 0 \iff p(b) = (b - b_1)(b - b_2) = 0 \implies \]
\[ p(1) = (1 - b_1)(1 - b_2) = 1 - \text{trace} + \det \]
\[ = -(1 - \alpha) \beta \bar{c} A \alpha k^{\alpha - 2} < 0 \]

This can only be true if one root (say \( b_1 \)) is less than 1 and the other root is greater than 1. We can then conclude (and confirm the predictions of the PD) that the equilibrium is \textit{saddle-point}. 

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After obtaining $b_1 < 1$ and $b_2 > 1$, we can have the general solution for this system:

\[
\begin{bmatrix}
\tilde{c}_t \\
\tilde{k}_t
\end{bmatrix} = \begin{bmatrix}
k_1 A_1 b_1^t + k_2 A_2 b_2^t \\
A_1 b_1^t + A_2 b_2^t
\end{bmatrix}.
\]  

(59)

where $k_1$ and $k_2$ are two constants determined by the roots (here we ignore their detailed values). Given $k_0$,

\[
\tilde{k}_0 = k_0 - \bar{k} = A_1 + A_2 \\
\tilde{c}_0 = c_0 - \bar{c} = k_1 A_1 + k_2 A_2
\]

Note that $b_2^t \to \infty$ as $t \to \infty$ because $b_2 > 1$. To guarantee a convergent time path to the I.E. (i.e., to kill the explosive path), the endogenous $\tilde{c}_0$ need to be set in the right way and make $A_2$ be 0:

\[
c_0 - \bar{c} = k_1 (k_0 - \bar{k}).
\]

(60)

Hence, given any $k_0$, we can find $c_0$ s.t. the economy “jump” to the pair of stable branches and then move to the saddle point equilibrium.